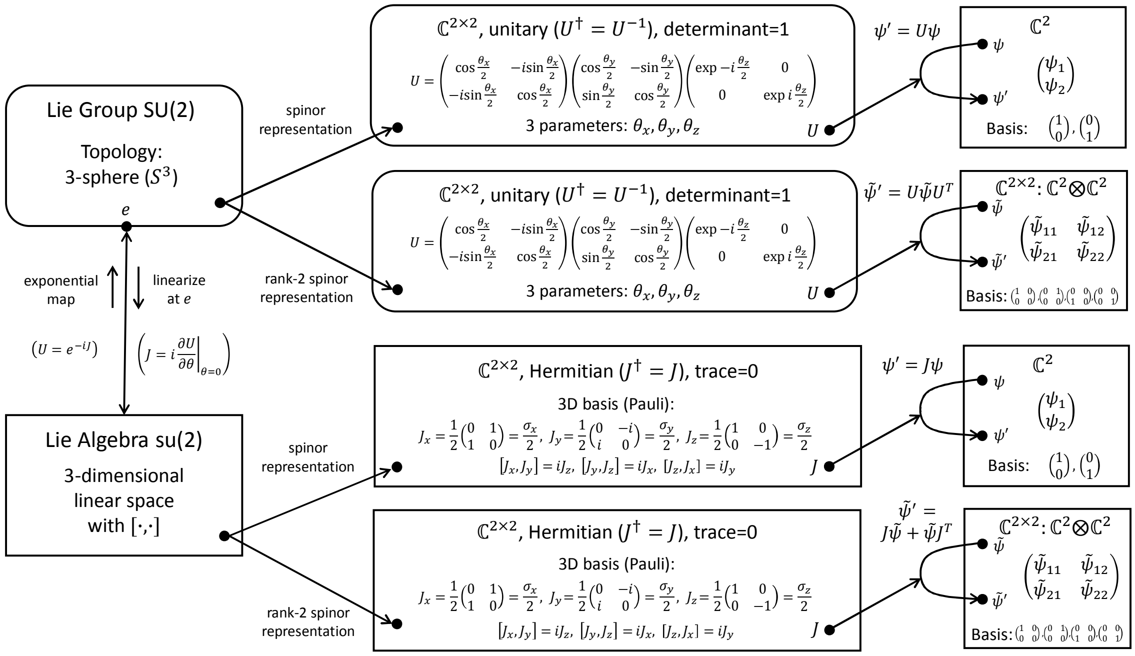
3.15 SU(2): Tensor-Product Representation; Two Spin-½ Particles and Entanglement



Let's switch gears and move on to another type of representation: the *tensor-product representation*. If we have two k-dimensional representations on vectors, we can form the tensor product of the two representation spaces and get a $(k \times k)$-dimensional representation on (rank-2) tensors. In this example, we take two copies of the 2-dimensional representation of SU(2), which act on spinors, and construct a new 4-dimensional representation, which acts on so-called *rank-2 spinors* or *2-index spinors* [RtR, Ch. 22.8; PfS, Ch. 3.7.8].

The vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ form a basis for the 2-dimensional vector space $\mathbb{C}^2$. To construct a basis for the tensor-product space $\mathbb{C}^2\otimes\mathbb{C}^2$, we take the tensor product (= outer product) of all possible basis-vector pairs: $\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}^T = \begin{pmatrix}1&0\\0&0\end{pmatrix}$, $\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}^T = \begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}^T = \begin{pmatrix}0&0\\1&0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}^T = \begin{pmatrix}0&0\\0&1\end{pmatrix}$. Given this tensor basis, a general element of the 4-dimensional tensor-product space can be written as $\tilde{\psi} = \begin{pmatrix}\tilde{\psi}_{11}&\tilde{\psi}_{12}\\\tilde{\psi}_{21}&\tilde{\psi}_{22}\end{pmatrix}$, just like a general element of the spinor space can be written as $\psi = \begin{pmatrix}\psi_1\\\psi_2\end{pmatrix}$. Mathematically, the object $\tilde{\psi}$ is a tensor, but in this context, it is usually called a rank-2 spinor.

It is important to distinguish between the *Cartesian product* and the *tensor product*. Whereas the Cartesian-product space, $\mathbb{C}^2 \times \mathbb{C}^2$, consists of *pairs* of spinors, the tensor-product space, $\mathbb{C}^2\otimes\mathbb{C}^2$, consists of rank-2 spinors. Not all rank-2 spinors can be expressed as the (tensor) product of spinor pairs, as we will see momentarily. In fact, given a k-dimensional vector space, the Cartesian-product space is $2k$ dimensional whereas the tensor-product space is k^2 dimensional! (For $k = 2$, as in our example, they both happen to be four dimensional.)

In quantum mechanics, a spinor describes the state of a spin-½ particles. What then do rank-2 spinors describe? They describe the *combined* state of *two* spin-½ particles! Let's make some examples. For each individual particle we take the state $\psi = \begin{pmatrix}1\\0\end{pmatrix}$ to mean "spin up" and the state $\psi = \begin{pmatrix}0\\1\end{pmatrix}$ to mean "spin down". Then, the combined state $\tilde{\psi} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix}^T$ simply means that both particles are "spin up". Next, what does the state $\tilde{\psi} = \frac{1}{2}\begin{pmatrix}1 & i\\-1 & -i\end{pmatrix}$ represent? This looks messy, but it can be written as the tensor product of two spinors: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}^T$. It is a so-called *product state*. We can interpret this combined state as one particle being in the superposition $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ and the other particle being in the superposition $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$. Each particle is in its own superposition of up and down. Finally, let's look at the 2-particle state $\tilde{\psi} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$. This state *cannot* be written as a product of two spinors, only as the *sum* of products: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} \cdot \begin{pmatrix}0\\1\end{pmatrix}^T - \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix}^T$. It is a so-called *entangled state*! The interpretation of this state is that the spins of the two particles are always opposite, but we don't know *anything* about each particle's spin quantum state [TM, Vol. II, Ch. 6]. Much of quantum mechanics' weirdness comes from the existence of such entangled states. Note that it is these entangled states that makes it necessary to use the tensor-product space to represent multi-particle states. If only product states existed, each particle could be described by its own state.

Now that we have a new representation space, we need to work out how the Lie-group elements act on it. We know how to construct a basis for rank-2 spinors from the basis of spinors and we know how to transform spinors, that's all we need! Rank-2 basis spinors are constructed from two basis spinors, ψ and ϕ, by taking the tensor product $\tilde{\psi} = \psi\phi^T$ and the basis spinors transform like $\psi' = U\psi$ and $\phi' = U\phi$. Thus, rank-2 basis spinors transform like $\tilde{\psi}' = U\psi(U\phi)^T = U\psi\phi^T U^T = U\tilde{\psi}U^T$. If we define the operator $\tilde{U}$ to act on rank-2 spinors like $\tilde{U}\tilde{\psi} := U\tilde{\psi}U^T$, we can simply write $\tilde{\psi}' = \tilde{U}\tilde{\psi}$. We can think of this transformation as follows: first U acts on all the column vectors of $\tilde{\psi}$ and then it acts on all the row vectors of the result, $U\tilde{\psi}U^T = (U(U\tilde{\psi})^T)^T$ (reversing the order, $U\tilde{\psi}U^T = U(U\tilde{\psi}^T)^T$, yields the same result). Although we derived this transformation for rank-2 basis spinors, it works for *any* rank-2 spinor that can be expressed in this basis (see the lower branch of the diagram).

How do the Lie-algebra elements of this new representation act on rank-2 spinors? To find out, we take the transformation law from above and write it as a function of the rotation angle θ about a particular axis, $U(\theta)\tilde{\psi}U(\theta)^T$, take the derivative with respect to θ, and set $\theta = 0$. Using the product rule, we find $J\tilde{\psi}U(0)^T + U(0)\tilde{\psi}J^T = J\tilde{\psi} + \tilde{\psi}J^T$, where J is the generator of $U(\theta)$ from the spinor representation (see the lower branch of the diagram). Finally, we can define a new operator $\tilde{J}$ that acts on rank-2 spinors like $\tilde{J}\tilde{\psi} := J\tilde{\psi} + \tilde{\psi}J^T$.

In quantum mechanics, the operator $\tilde{J}$ represents the observable for the *combined* spin of our two-spin system (along a given axis). To understand this operator better, let's act with it on the product state $\tilde{\psi} = \psi\phi^T$, where ψ is the state of a first particle and ϕ is the state of a second particle: $\tilde{J}\tilde{\psi} = J\tilde{\psi} + \tilde{\psi}J^T = J\psi\phi^T + \psi\phi^T J^T = (J\psi)\phi^T + \psi(J\phi)^T$. Note that the first part of the operator acts only on the part of the state associated with the first particle, ψ, and the second part acts only on the part of the state associated with the second particle, ϕ. If the two states are eigenstates of J with eigenvalues j_ψ and j_ϕ, that is, $J\psi = j_\psi\psi$ and $J\phi = j_\phi\phi$, then he have $\tilde{J}\psi\phi^T = ((j_\psi\psi)\phi^T + \psi(j_\phi\phi)^T = (j_\psi + j_\phi)\psi\phi^T$, which means that the eigenvalue of $\tilde{J}$ is $j_\psi + j_\phi$. In this sense, we can think of the combined spin as the sum of the two constituent spins.