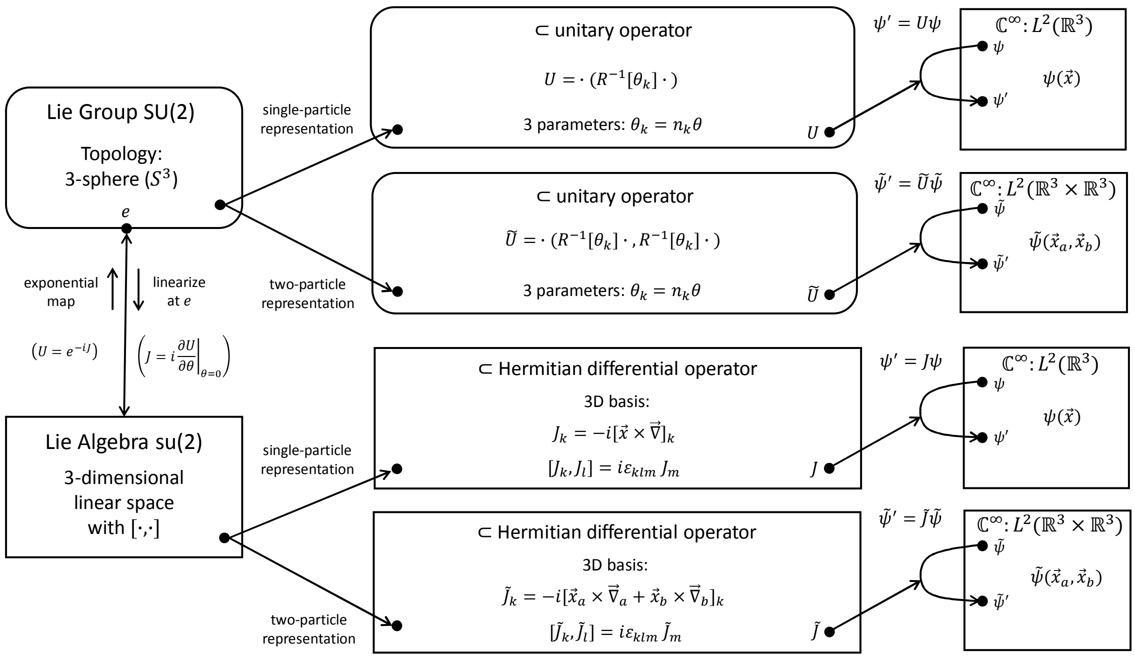
3.19 SU(2): Representation on Two-Particle Wave Functions



After discussing two spin-½ particles with fixed positions, we now turn to two particles that can move around freely (ignoring the spin in this example). To describe such a two-particle system we need a *wave function* that depends on the positions of both particles.

The upper branch of the diagram shows the familiar representation on single-particle wave functions. The representation space consists of the square-integrable complex functions of three real variables, $L^2(\mathbb{R}^3)$. For a two-particle system (lower branch of the diagram), the representation space is the *tensor product* of two copies of this space, $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$. It turns out that this is the same as the space of square-integrable complex functions of *six* real variables: $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3) = L^2(\mathbb{R}^3 \times \mathbb{R}^3) = L^2(\mathbb{R}^6)$ [QTGR, Ch. 9.1]. The six variables represent the 3D position coordinates of particles a and b: $(\vec{x}_a, \vec{x}_b)$. A point in this 6-dimensional space describes a so-called *configuration* of the system. It is essential to realize that the wave function of a two-particle system, $\tilde{\psi}(\vec{x}_a, \vec{x}_b)$, is defined on a 6-dimensional configuration space, *not* on ordinary physical space.

As pointed out earlier, it is important to distinguish between the Cartesian product and the tensor product. Whereas the Cartesian-product space, $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, consists of *pairs of wave functions*, $(\psi_a(\vec{x}), \psi_b(\vec{x}))$, the tensor-product space, $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$, consists of wave functions that depend on a *pair of arguments*, $\tilde{\psi}(\vec{x}_a, \vec{x}_b)$. There are many functions $\tilde{\psi}(\vec{x}_a, \vec{x}_b)$ that *cannot* be expressed as the product of two functions $\psi_a(\vec{x}_a) \cdot \psi_b(\vec{x}_b)$. In other words, only some two-particle wave functions can be written as a product of two single-particle wave functions, while most two-particle wave functions cannot be split up in this way. If a wave function can be split up, it is a *product wave function*; if it cannot be split up, it is an *entangled wave function*. In the first case, the two particles can be treated as independent entities described by two wave functions in physical space that look just like classical fields; in the second case, the two particles are quantum-mechanically linked together and must be described by a *single* wave function. All this is completely analogous to what we have said earlier about two spin-½ particles with fixed positions.

A simple example of a product wave function is $\tilde{\psi}(\vec{x}_a, \vec{x}_b) = e^{i(\vec{k}_a \cdot \vec{x}_a)} e^{i(\vec{k}_b \cdot \vec{x}_b)}$. The first factor describes a (static) plane wave oriented in the direction $\vec{k}_a$ and the second factor describes another plane wave oriented in the direction $\vec{k}_b$. Quantum mechanically, the wave function represents one particle with momentum $\vec{k}_a$ and another particle with momentum $\vec{k}_b$ (in units for which $\hbar = 1$). Now, let's consider the superposition $\tilde{\psi}(\vec{x}_a, \vec{x}_b) = \sum_{\vec{k}} e^{i(\vec{k} \cdot \vec{x}_a)} e^{i(-\vec{k} \cdot \vec{x}_b)}$, where the sum goes over all possible directions $\vec{k}$. This is *no* longer a product wave function! It represents two particles going in opposite directions with the same absolute momentum, where all directions are *equally likely*. Think of a particle at rest that decays into two particles of equal mass. Although we know the two-particle wave function exactly, we don't know anything about the individual particles: we are dealing with an entangled wave function! (Note how a probabilistic theory that conserves momentum forces entanglement on us!)

How does our two-particle wave function transform under SU(2)? We know that a single-particle wave function transforms like $\psi'(\vec{x}) = \psi(R^{-1}\vec{x})$, where R is the 3D rotation matrix. A product wave function of two particles thus transforms like $\tilde{\psi}'(\vec{x}_a, \vec{x}_b) = \psi_a(R^{-1}\vec{x}_a) \cdot \psi_b(R^{-1}\vec{x}_b)$. We therefore expect a general two-particle wave function to transform like $\tilde{\psi}'(\vec{x}_a, \vec{x}_b) = \tilde{\psi}(R^{-1}\vec{x}_a, R^{-1}\vec{x}_b)$. Splitting off the transformation from the wave function, we can write $\tilde{U} = \cdot\,(R^{-1}\ \cdot, R^{-1}\ \cdot)$, as shown in the lower branch of the diagram. This makes intuitive sense: we simply rotate the position coordinates of both particles in the same way.

What are the basis generators of this representation? A rotation about the z-axis by the angle θ_z can be written explicitly as

$$\tilde{U}(\theta_z)\, \tilde{\psi}(\vec{x}_a, \vec{x}_b) = \tilde{\psi}\left(\begin{bmatrix} x_{1a}\cos\theta_z + x_{2a}\sin\theta_z \\ -x_{1a}\sin\theta_z + x_{2a}\cos\theta_z \\ x_{3a} \end{bmatrix}, \begin{bmatrix} x_{1b}\cos\theta_z + x_{2b}\sin\theta_z \\ -x_{1b}\sin\theta_z + x_{2b}\cos\theta_z \\ x_{3b} \end{bmatrix}\right).$$

Taking the derivative with respect to θ_z, setting θ_z to zero, and multiplying by i yields the generator $\tilde{J}_z$ acting on the wave function:

$$\tilde{J}_z\, \tilde{\psi}(\vec{x}_a, \vec{x}_b) = i\left(\frac{\partial\tilde{\psi}}{\partial x_{1a}} \cdot x_{2a} + \frac{\partial\tilde{\psi}}{\partial x_{2a}} \cdot (-x_{1a}) + \frac{\partial\tilde{\psi}}{\partial x_{1b}} \cdot x_{2b} + \frac{\partial\tilde{\psi}}{\partial x_{2b}} \cdot (-x_{1b})\right).$$

Splitting off the operator from the wave function, we get the basis generator

$$\tilde{J}_z = -i\left(x_{1a}\frac{\partial}{\partial x_{2a}} - x_{2a}\frac{\partial}{\partial x_{1a}} + x_{1b}\frac{\partial}{\partial x_{2b}} - x_{2b}\frac{\partial}{\partial x_{1b}}\right) = -i\left([\vec{x}_a \times \vec{\nabla}_a]_z + [\vec{x}_b \times \vec{\nabla}_b]_z\right),$$

where the second form uses the z components of two different cross products. The other two basis generators can be found in the same way and are just the x and y components of the same cross products (see the diagram). In quantum mechanics, these generators correspond to the observables for the *combined* orbital angular momentum of the two-particle system. Each operator has two parts, which we can associate with the angular momenta of particles a and b.