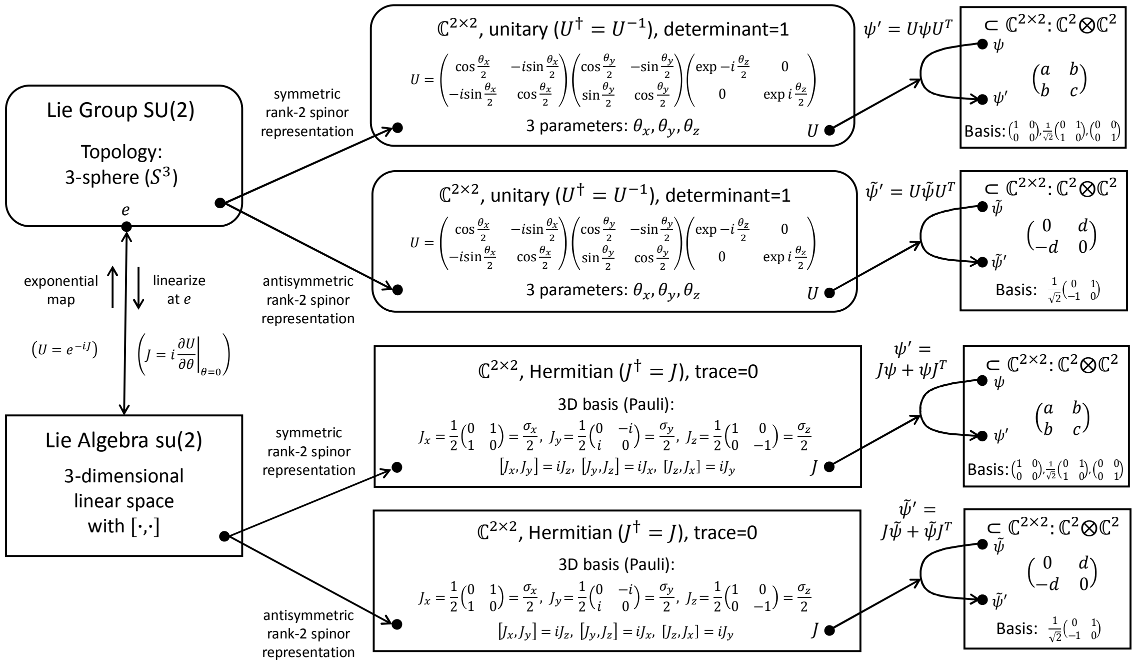
3.17 SU(2): Representations on Symmetric and Antisymmetric Rank-2 Spinors



In the previous example we found a useful basis for our 4-dimensional representation space. Now reverting from 4D vectors back to rank-2 spinors, this basis is: $\Psi_{0,0} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Psi_{1,+1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\Psi_{1,0} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\Psi_{1,-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We realized with the help of the Casimir operator that this 4-dimensional representation of SU(2) breaks up into a 1-dimensional and a 3-dimensional irreducible representation. In the following, we use a different approach to do the same thing.

Any rank-2 tensor can be written as the sum of a symmetric and an antisymmetric part. For the case of a rank-2 spinor, we have $\begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2\psi_{11} & \psi_{12} + \psi_{21} \\ \psi_{21} + \psi_{12} & 2\psi_{22} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \psi_{12} - \psi_{21} \\ \psi_{21} - \psi_{12} & 0 \end{pmatrix}$, where the first matrix on the right-hand side is symmetric and the second one is antisymmetric. Note that the symmetric matrix has three independent components while the antisymmetric matrix has only one. Now, it turns out that any transformation of the form $\psi' = U\psi U^T$ maps a symmetric matrix ψ to a symmetric matrix ψ' and an antisymmetric matrix to an antisymmetric matrix (U does not even have to be unitary). The two types of matrices don't mix! Thus, our rank-2 spinor representation must break up into a 3-dimensional (symmetric) and a 1-dimensional (antisymmetric) representation.

It won't come as a surprise that the two representations are just slightly disguised versions of the 1- and 3-dimensional representations that we discussed earlier. The *antisymmetric* representation is shown in the lower branch of the diagram. If it behaves like the trivial (1-dimensional) representation, then any generator acting on an antisymmetric rank-2 spinor must result in zero, corresponding to no change (= identity transformation). Picking the generator $J_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and acting with it on the antisymmetric rank-2 spinor $\tilde{\psi} = z\,\Psi_{0,0} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}$, where z is a complex coefficient, yields $\tilde{\psi}' = J_z\tilde{\psi} + \tilde{\psi}J_z^T =$

$\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0 & z\\ -z & 0\end{pmatrix}+\frac{1}{\sqrt{2}}\begin{pmatrix}0 & z\\ -z & 0\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$, as expected. Acting with J_x or J_y on $\tilde{\psi}$ also yields zero.

Next, let's act with a generator on a *symmetric* rank-2 spinor (see the upper branch of the diagram). Such a spinor can be written as $\psi = z_1\Psi_{1,+1} + z_2\Psi_{1,0} + z_3\Psi_{1,-1} = \begin{pmatrix} z_1 & z_2/\sqrt{2}\\ z_2/\sqrt{2} & z_3\end{pmatrix}$, where z_1, z_2, z_3 are three complex coefficients. Acting with J_z on this rank-2 spinor yields $\psi' = J_z\psi + \psi J_z^T = \begin{pmatrix} z_1 & 0\\ 0 & -z_3\end{pmatrix}$. Comparing with $\psi' = \begin{pmatrix} z_1' & z_2'/\sqrt{2}\\ z_2'/\sqrt{2} & z_3'\end{pmatrix}$, we find that $z_1' = z_1$, $z_2' = 0$, and $z_3' = -z_3$. Thus, the unpacked generator is exactly the 3×3 matrix we had before when rotating spin-1 particles about the z axis. Unpacking the remaining two generators confirms that this representation is equivalent to the 3-dimensional representation that we discussed earlier.

As we have seen, the tensor product of two copies of the 2-dimensional representation of SU(2) breaks up into a 3-dimensional and a 1-dimensional irreducible representation. This fact can be expressed as a formula: $\mathbf{2}\otimes\mathbf{2} = \mathbf{3}\oplus\mathbf{1}$, where bold numbers indicate the dimension of the representation, $\otimes$ stands for the *tensor product*, and $\oplus$ stands for the *direct sum* of two representations. The direct sum of two vector spaces consists of the Cartesian product of the sets together with the operations of vector addition and scalar multiplication. The dimension of the resulting vector space is the *sum* of the two constituent spaces (in contrast to the tensor product for which it is the product).

To construct new irreducible representations from old ones, we can take the tensor product of two known representations and break the result up into irreducible representations. We already know that $\mathbf{2}\otimes\mathbf{2} = \mathbf{3}\oplus\mathbf{1}$. Similarly, it can be shown that $\mathbf{2}\otimes\mathbf{3} = \mathbf{4}\oplus\mathbf{2}$, $\mathbf{2}\otimes\mathbf{4} = \mathbf{5}\oplus\mathbf{3}$, and $\mathbf{3}\otimes\mathbf{3} = \mathbf{5}\oplus\mathbf{3}\oplus\mathbf{1}$. The total number of dimensions on both sides of the equal sign is always the same: $2\times 3 = 6 = 4+2$, $2\times 4 = 8 = 5+3$, and $3\times 3 = 9 = 5+3+1$. The process of breaking up tensor-product representations into irreducibles is known as *Clebsch-Gordan decomposition* [GTNut, Ch. IV.3].

We can also take the tensor product of more than two representations, resulting in a representation on higher-rank tensors. Specifically, we can take the tensor product of k copies of the 2-dimensional (spinor) representation to get a representation on rank-k spinors. Then, we break this 2^k-dimensional representation into irreducibles. For example, for $k=2$: $\mathbf{2}\otimes\mathbf{2} = \mathbf{3}\oplus\mathbf{1}$, as we already know, for $k=3$: $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2} = \mathbf{2}\otimes(\mathbf{3}\oplus\mathbf{1}) = (\mathbf{2}\otimes\mathbf{3})\oplus(\mathbf{2}\otimes\mathbf{1}) = \mathbf{4}\oplus\mathbf{2}\oplus\mathbf{2}$, and for $k=4$: $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2} = \mathbf{2}\otimes(\mathbf{4}\oplus\mathbf{2}\oplus\mathbf{2}) = (\mathbf{2}\otimes\mathbf{4})\oplus(\mathbf{2}\otimes\mathbf{2})\oplus(\mathbf{2}\otimes\mathbf{2}) = \mathbf{5}\oplus\mathbf{3}\oplus\mathbf{3}\oplus\mathbf{3}\oplus\mathbf{1}\oplus\mathbf{1}$.

Interestingly, we can obtain *all* irreducible representations of SU(2) in this way: as we step through $k = 2, 3, 4, \ldots$, we get a new $(k+1)$-dimensional irreducible representation at every step! For example, we get $\mathbf{3}$ from $\mathbf{2}\otimes\mathbf{2}$, we get $\mathbf{4}$ from $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}$, and we get $\mathbf{5}$ from $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}$. It turns out that this new irreducible representation is furnished by the *totally symmetric* rank-k spinors [GTNut, Ch. IV.5; RtR, Ch. 22.8]. Thus, the n-dimensional representation of SU(2) can act not only on n-dimensional vectors but also on totally symmetric rank-$(n-1)$ spinors (e.g., the 3-dimensional representation can also act on symmetric rank-2 spinors). As a consequence, the quantum state of a particle with total spin j can be described not only by a vector with $2j+1$ components but also by a totally symmetric spinor of rank $2j$. This new point of view leads to an interesting geometric interpretation of spin states known as the *Majorana picture* (see [RtR, Ch. 22.10] for details).