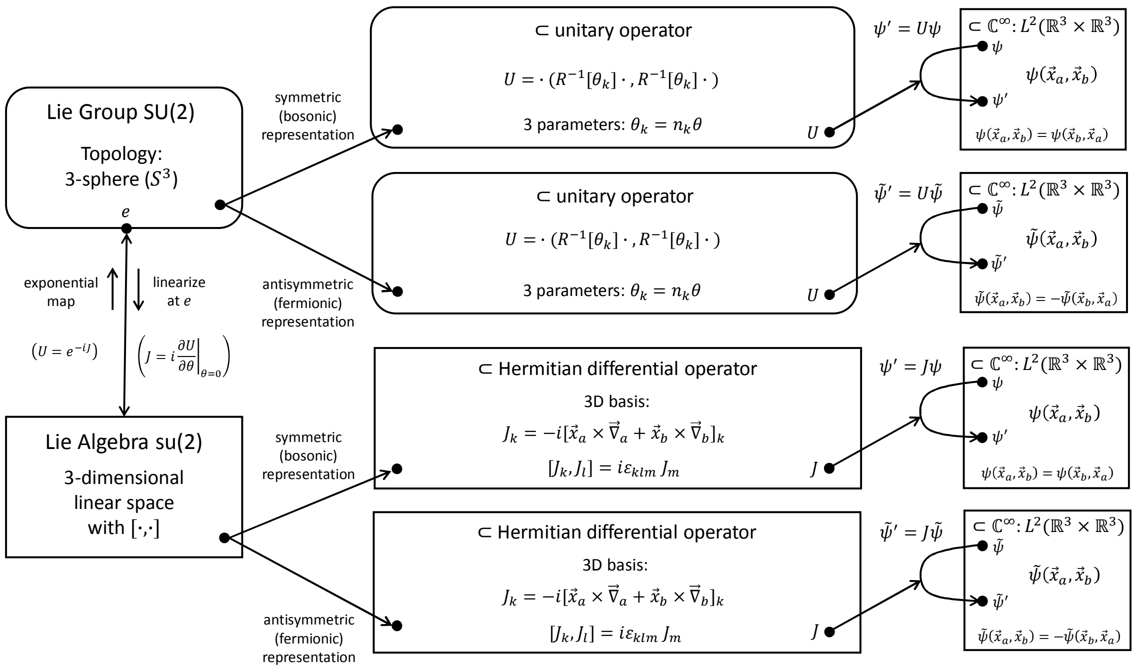
3.20 SU(2): Representations on (Anti)Symmetric Functions; Bosons and Fermions



The function from the previous example, $\psi(\vec{x}_a, \vec{x}_b)$, can be decomposed into a part that is *symmetric* with respect to an exchange of its arguments $\vec{x}_a$ and $\vec{x}_b$ and a part that is *antisymmetric*: $\psi(\vec{x}_a, \vec{x}_b) = \psi_S(\vec{x}_a, \vec{x}_b) + \psi_A(\vec{x}_a, \vec{x}_b)$, where $\psi_S(\vec{x}_a, \vec{x}_b) = \psi_S(\vec{x}_b, \vec{x}_a)$ and $\psi_A(\vec{x}_a, \vec{x}_b) = -\psi_A(\vec{x}_b, \vec{x}_a)$. The first part is obtained by symmetrizing and the second part by antisymmetrizing the original function: $\psi_S(\vec{x}_a, \vec{x}_b) = \frac{1}{2}[\psi(\vec{x}_a, \vec{x}_b) + \psi(\vec{x}_b, \vec{x}_a)]$ and $\psi_A(\vec{x}_a, \vec{x}_b) = \frac{1}{2}[\psi(\vec{x}_a, \vec{x}_b) - \psi(\vec{x}_b, \vec{x}_a)]$. Any transformation of the form $\psi'(\vec{x}_a, \vec{x}_b) = \psi(R\vec{x}_a, R\vec{x}_b)$ maps symmetric functions ψ_S to symmetric functions ψ'_S and antisymmetric functions ψ_A to antisymmetric functions ψ'_A (R does not even have to be a rotation). Analogous to what we have seen for rank-2 spinors, the symmetric and the antisymmetric parts don't mix! Thus, our representation breaks up into two sub-representations (see the diagram, where we named the symmetric and antisymmetric functions ψ and $\tilde{\psi}$, respectively, to reduce clutter). In contrast to the rank-2 spinors, where the sub-representations were one and three dimensional and irreducible, here the sub-representations are *infinite dimensional* and *reducible*.

What is the quantum-mechanical interpretation of symmetric and antisymmetric wave functions? Both types of wave functions can be decomposed into basis functions that represent particle pairs with a joint orbital angular momentum given by $j = 0, 1, 2, \ldots$ and $m = 0, \pm 1, \pm 2, \ldots \pm j$. The restriction on the wavefunction is not related to orbital angular momentum but to the *type* of particles: our new representations describe systems of two *identical* (= *indistinguishable*) particles! The defining property of two identical particles is that we can exchange them without any observable effects. Indeed, the observable *probability-density* function does not change under such an exchange: $|\psi_S(\vec{x}_a, \vec{x}_b)|^2 = |\psi_S(\vec{x}_b, \vec{x}_a)|^2$ as well as $|\psi_A(\vec{x}_a, \vec{x}_b)|^2 = |\psi_A(\vec{x}_b, \vec{x}_a)|^2$. Surprisingly, we found *two* types of wave functions for which this is true and thus there are *two* types of identical particles! Those with a symmetric wave function are called *bosons* and those with an antisymmetric wave function are called *fermions*. Rotations (as well as many other transformations) always map bosons to bosons and fermions to fermions.

Some symmetric (bosonic) two-particle wave functions can be written as a product of two identical single-particle wave functions: $\psi_S(\vec{x}_a, \vec{x}_b) = \phi(\vec{x}_a)\phi(\vec{x}_b)$. Such a product wave function can be thought of as a field $\phi(\vec{x})$ living in *physical space* and, when generalized to many identical bosons, behaves like a classical field. This is the origin of large-scale quantum effects occurring, for example, for laser light, superfluids, and superconductor. A more general symmetric two-particle wave function can be written as the symmetrized product of two *different* single-particle wave functions $\psi_S(\vec{x}_a, \vec{x}_b) = \phi(\vec{x}_a)\chi(\vec{x}_b) + \chi(\vec{x}_a)\phi(\vec{x}_b)$. This is no longer a product wave function and thus describes two *entangled* bosons. However, the entanglement is minimal in the sense that the wave function can be *split* into two subfunctions. The most general symmetric two-particle wave function can be written as $\psi_S(\vec{x}_a, \vec{x}_b) = \sum_{i=1}^{n}[\phi_i(\vec{x}_a)\chi_i(\vec{x}_b) + \chi_i(\vec{x}_a)\phi_i(\vec{x}_b)]$ and is *non-splitting* (for $n > 1$) [RtR, Ch. 23.8].

An antisymmetric (fermionic) two-particle wave function can never be written as a product of two single-particle wave function. This is the origin of the *Pauli exclusion principle*. Some antisymmetric two-particle wave functions can be written as $\psi_A(\vec{x}_a, \vec{x}_b) = \phi(\vec{x}_a)\chi(\vec{x}_b) - \chi(\vec{x}_a)\phi(\vec{x}_b)$, that is, the antisymmetrized product of two different single-particle wave functions. A general antisymmetric two-particle wave function can be written as $\psi_A(\vec{x}_a, \vec{x}_b) = \sum_{i=1}^{n}[\phi_i(\vec{x}_a)\chi_i(\vec{x}_b) - \chi_i(\vec{x}_a)\phi_i(\vec{x}_b)]$. Thus, fermions are *always* entangled. There is minimum amount of entanglement (given by the first form) that cannot be avoided. In particular, all electrons in the universe are entangled with one other: we cannot really describe individual electrons by single-particle wave functions!

So, what do we mean then when we say that one electron in an atom occupies the *1s* orbital and another one a *2p* orbital? This is a sloppy way of speaking! What we really mean is that the wave function for all the electrons in the atom is the antisymmetrized product of a wave function for the *1s* orbital, a wave function for the *2p* orbital, etc. Similarly, when we say that no two electrons can occupy the same state, what we really mean is that a two-particle wave function that is the antisymmetrized product of two identical single-particle wave functions is zero, that is, it can't exist.

For another perspective on identical particles, let's assume that our particles can only be either at location 1 or location 2 [RtR, Ch. 23.7]. Then, the single-particle wave function becomes a two-component vector, ψ_i where $i = 1, 2$, and the two-particle wave function becomes a 2×2 tensor, ψ_{ij} where $i, j = 1, 2$. The tensor component ψ_{12}, for example, is the amplitude for particle a to be at location 1 and particle b to be at location 2. Now, if the two particles are identical, $|\psi_{ij}|^2 = |\psi_{ji}|^2$ must hold and when further requiring that two successive particle exchanges leave the state unchanged, we get either $\psi_{ij} = \psi_{ji}$ (symmetric state for bosons) or $\psi_{ij} = -\psi_{ji}$ (antisymmetric state for fermions).

Some of these symmetric states are product states of the form $\psi_{ij} = \phi_i\phi_j$, which means that both bosons are in the same state ϕ. In other words, either the bosons are both at location 1 or they are both at location 2 or they are in the same superposition of being at locations 1 and 2. A more general symmetric state is of the form $\psi_{ij} = \phi_i\chi_j + \chi_i\phi_j$, which is an entangled state. For example, one boson is at location 1 and the other one at location 2, but we don't know which one is where. Antisymmetric states can never be product states. Some antisymmetric states are of the form $\psi_{ij} = \phi_i\chi_j - \chi_i\phi_j$, which is again an entangled state. For example, one fermion is at location 1 and the other one at location 2, but we don't know which one is where.