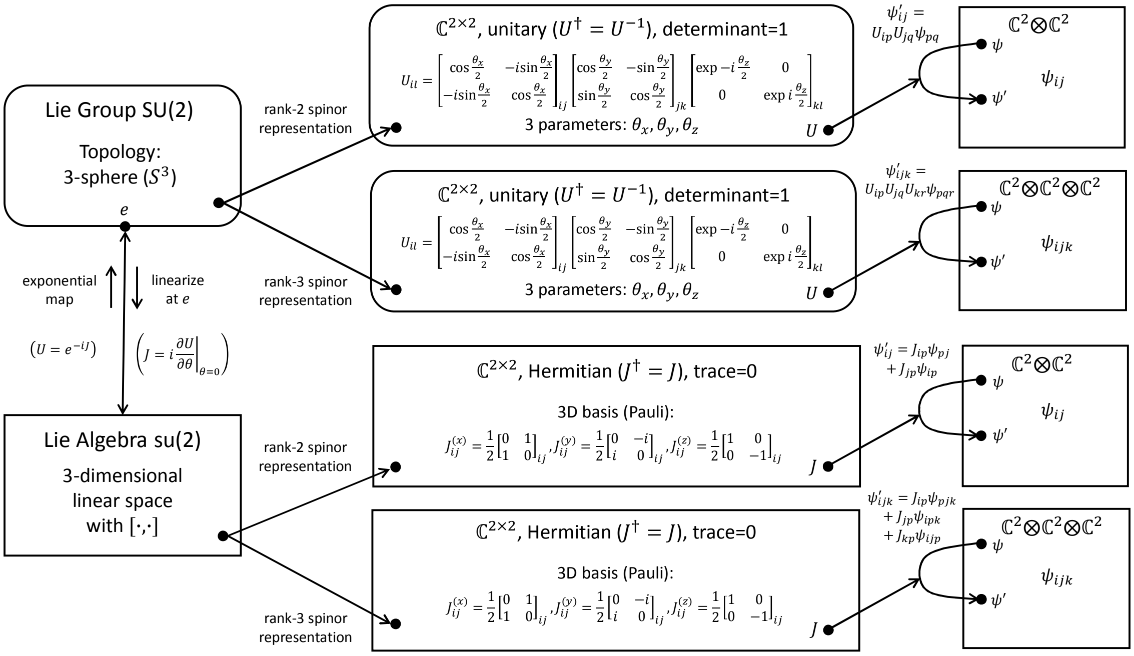
3.18 SU(2): Representation on Rank-3 Spinors; Index Notation



Having discussed the tensor-product representation $\mathbf{2}\otimes\mathbf{2}$, we now turn to the 8-dimensional $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}$ representation of SU(2). In quantum mechanics, this representation describes transformations of the state of *three* (possibly entangled) spin-½ particles. Now, the elements of the representation space are *rank-3 spinors*, which we can think of as 2×2×2 cubic number schemes. How do we deal with these unwieldy objects?

One way to deal with rank-3 tensors is to write them as column vectors where each component is a matrix. A better way, which generalizes to any rank, is to introduce the index notation along with the Einstein summation convention. How does this work? We used to write the product of a matrix U with a vector ϕ as $U\phi$. Now, we write the matrix as U_{ij}, the vector as ϕ_i, and their product as $U_{ij}\phi_j$, where a summation over the repeated index j is implied. The transformation of a matrix ψ, which we used to write as $\psi' = U\psi U^T$, we now write as $\psi'_{ij} = U_{ip}\psi_{pq}U^T_{qj} = U_{ip}U_{jq}\psi_{pq}$, where summation over the two repeated indices, p and q, is implied. The identity matrix, which we used to write as I, we now write as the Kronecker delta δ_{ij}. With this new notation in hand, we can write a rank-3 tensor as a symbol with three indices: ψ_{ijk}!

The upper branch of the diagram shows the representation on rank-2 spinors, which we are already familiar with, but now rewritten using the index notation. The Lie-group action $\psi' = U\psi U^T$ becomes $\psi'_{ij} = U_{ip}U_{jq}\psi_{pq}$ (as already mentioned) and the Lie-algebra action $\psi' = J\psi + \psi J^T$ becomes $\psi'_{ij} = J_{ip}\psi_{pj} + \psi_{ip}J^T_{pj} = J_{ip}\psi_{pj} + J_{jp}\psi_{ip}$. To write down the basis generators in our new notation, we need to make a small change. We used to label the basis generators as J_x, J_y, and J_y, but now the x, y, z indices are in conflict with the new indices for labeling the components. To resolve this issue, we move the x, y, z indices "upstairs" and put them in parenthesis to avoid confusion with an exponent. In our new notation, the basis generators are written as $J^{(x)}_{ij}$, $J^{(y)}_{ij}$, and $J^{(z)}_{ij}$.

Now that we are getting used to the index notation, let's tackle the rank-3 spinor ψ_{ijk} (see the lower branch of the diagram)! Analogous to what we did for the rank-2 spinor, we write the rank-3 spinor as a tensor product of three spinors. Then, we use our knowledge of how spinors transform to infer how the rank-3 spinor transforms. Writing the rank-3 spinor as the tensor product of three spinors, $\psi_{ijk} = \psi_i \phi_j \chi_k$, and then transforming the individual spinors yields $\psi'_{ijk} = U_{ip}\psi_p U_{jq}\phi_q U_{kr}\chi_r = U_{ip}U_{jq}U_{kr}\psi_p\phi_q\chi_r = U_{ip}U_{jq}U_{kr}\psi_{pqr}$, which is a straightforward generalization of the formula for rank-2 spinors. To find out how the Lie-algebra elements act on rank-3 spinors, we take the derivative of the transformation (using the product rule) and evaluate the result at the identity element, which yields $\psi'_{ijk} = [J_{ip}\delta_{jq}\delta_{kr} + \delta_{ip}J_{jq}\delta_{kr} + \delta_{ip}\delta_{jq}J_{kr}]\psi_{pqr}$. We can simplify this expression by eliminating the Kronecker deltas and renaming spinor indices as necessary: $\psi'_{ijk} = J_{ip}\psi_{pjk} + J_{jp}\psi_{ipk} + J_{kp}\psi_{ijp}$. Compared to the formula for rank-2 spinors, we now have three instead of two terms.

In quantum mechanics, the generator given by $J_{ip}\psi_{pjk} + J_{jp}\psi_{ipk} + J_{kp}\psi_{ijp}$ is the observable for the combined spin of a system of three spin-½ particles. Whereas for the 2-particle system the eigenvalues were in the set $\{\pm\frac{1}{2} \pm \frac{1}{2}\} = \{-1, 0, +1\}$, they are now in the set $\{\pm\frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2}\} = \{-\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}\}$. In other words, the possible measurement outcomes for the combined spin of the 3-particle system (along a given axis) are $-3/2$, $-1/2$, $+1/2$, or $+3/2$.

When we discussed rank-2 tensors we distinguished between symmetric and antisymmetric tensors. For higher-rank tensors, there are more possibilities! Rank-3 tensors can be symmetric (or antisymmetric) with respect to the first two indices, the last two indices, the first and last index, or all three indices. The last possibility is known as *totally* symmetric (or antisymmetric). Whereas a general rank-3 spinor has eight independent components (2×2×2), one that is symmetric with respect to two indices has only six, and one that is totally symmetric has only four. (Remember that totally symmetric rank-3 spinors furnish the 4-dimensional representation of SU(2)). A rank-3 spinor that is antisymmetric with respect to two indices has only two independent components and one that is totally antisymmetric must be zero. For a totally antisymmetric tensor to be nonzero, its dimension must be equal to or greater than its rank. For example, in three dimensions there is a totally antisymmetric rank-3 tensor that is nonzero: the Levi-Civita symbol ε_{ijk} (there are Levi-Civita symbols for every dimension = rank).

With our new index notation in hand, it is now easy to generalize the transformation law to an arbitrary rank: a rank-n tensor has n indices $\psi_{ijk\cdots m}$ and transforms like $\psi'_{ijk..m} = U_{ip}U_{jq}U_{kr}\cdots U_{mt}\psi_{pqr\cdots t}$. This formula makes clear that the transformation acts on each index independently. The indices live in the same household but do not talk to each other [GTNut, p. 188]! This is the reason why a tensor that is symmetric (or antisymmetric) with respect to a set of indices remains that way after a transformation.

You may have wondered in the previous example what the difference between the *direct sum* and the *direct product* is. In either case, the set structure is given by the Cartesian product. The difference is whether the algebraic operation on the set is called "sum" (e.g., for vector spaces) or "product" (e.g., for groups). We use the symbol $\times$ for the Cartesian product as well as for the direct product of groups and we use the symbol $\oplus$ for the direct sum of representations and algebras. It is important to realize that the tensor product, for which we us the symbol $\otimes$, is entirely different from the direct product! (See https://www.math3ma.com/blog/the-tensor-product-demystified.)