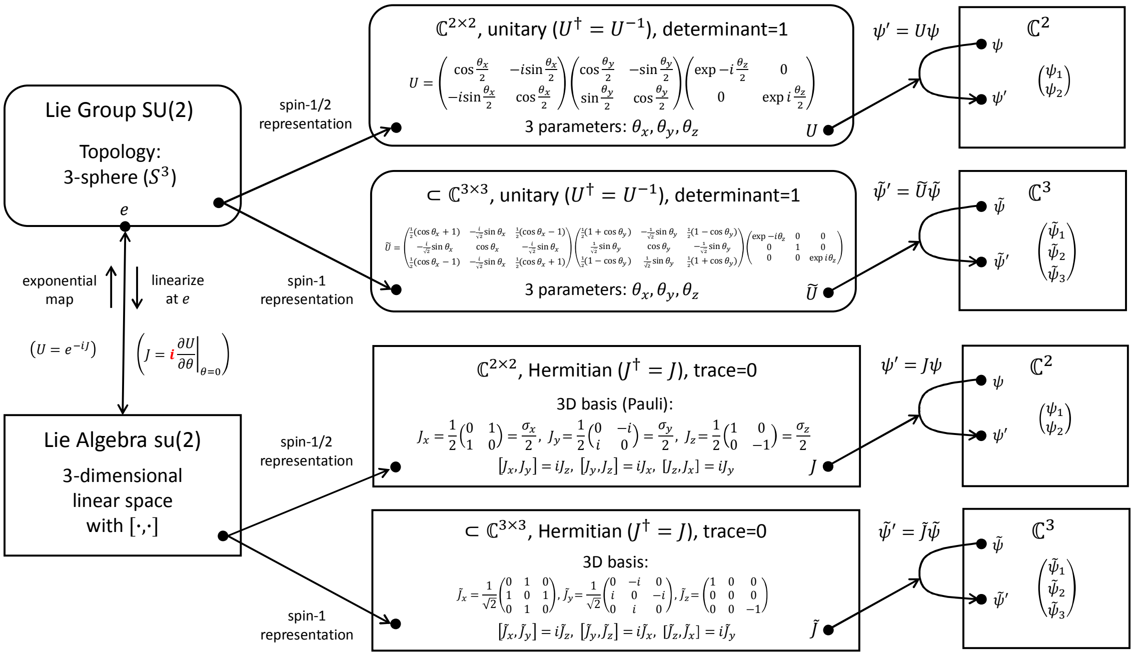
3.2 SU(2): Application to Quantum-Mechanical Spin



What is the group SU(2) good for? Unitary groups are tailor-made for quantum mechanics! In quantum mechanics the state of a system is represented by a complex vector. We will use the symbols ψ and ϕ for the column vectors representing such states (in Dirac notation they would be written as $|\psi\rangle$ and $|\phi\rangle$). The probability amplitude for one state being in another state is given by the Hermitian inner product $\phi^\dagger\psi$ (in Dirac notation: $\langle\phi|\psi\rangle$). By definition, unitary transformations satisfy $U^\dagger U = I$ and thus preserve the Hermitian inner product: $(U\phi)^\dagger U\psi = \phi^\dagger U^\dagger U\psi = \phi^\dagger\psi$. In particular, unitary transformations keep unambiguously distinguishable states for which $\phi^\dagger\psi = 0$ as such, thus preserving the information contained in the states [TM, Vol. 2]. Therefore, U(n) or SU(n) are exactly what we need for transforming quantum states! In the following, we focus on quantum-mechanical *spin states*, for which the three transformation parameters of SU(2) have a concrete geometrical meaning.

A spin-½ particle, such as an electron, is described by a state vector ψ with two complex components, a so-called *spinor.* (We assume that the electron's position is fixed so that we don't have to worry about that part of the state.) The components represent the amplitudes for *spin up* (spin +½) and *spin down* (spin −½) relative to a given direction in space, usually the z axis. Now, it turns out that the elements of the defining representation of SU(2) describe exactly how this state transforms under *3D rotation*: $\psi' = U(\theta_x, \theta_y, \theta_z)\psi$, where $\theta_x, \theta_y, \theta_z$ are the angles of rotation about the x, y, z axis, respectively. (See [FLP, Vol. III, Ch. 6] for more about these transformations.)

What about a (massive) spin-1 particle, such as a W or Z boson? Its spin state is described by a complex 3-component vector. Now, the components represent the amplitudes for *spin up* (spin +1), *spin horizontal* (spin 0), and *spin down* (spin −1). Remarkably, this state, $\tilde{\psi}$, also transforms under SU(2), but this time under its *3-dimensional representation*, as shown in the lower branch of the diagram. (See [FLP, Vol. III, Ch. 5] for more about these transformations.) In fact, SU(2) can rotate the spin state of any (massive) particle, we just need to use the appropriate representation!

The elements of the Lie group SU(2) tell us how to transform a spin state under rotation. What about the elements of the Lie algebra su(2)? Amazingly, when multiplied by $i\hbar$, they are the operators for the *angular-momentum* observable! More specifically, the operators for the *intrinsic* angular momentum or *spin*. Note that $\hbar$ has the units of angular momentum. To keep the notation simple, we will from now on work in units for which $\hbar = 1$. Moreover, physicists like to include a factor i as part of the generator (marked in red in the diagram) and then put a $-i$ into the exponent of the exponential map ($i \times -i = 1$) and an i into the commutation relations to keep things consistent. After these modifications, the generators become Hermitian (rather than anti-Hermitian) and their eigenvalues become real (rather than imaginary), as appropriate for observables. Because the generators now represent spin observables, we use the letter J (instead of X). We say, for example, that the basis generator J_z is the operator for the spin along the z-axis or, more simply, the z-component of the spin.

How do we use the operator J and the quantum state ψ to determine the observed value? The expectation value for a spin component is given by the "sandwich" formula $\langle J \rangle = \psi^\dagger J \psi$, where J is the generator matrix for the particular spin component (in Dirac notation: $\langle J \rangle = \langle \psi | J | \psi \rangle$). The full probability distribution is obtained by taking the Hermitian inner products of the state ψ and all the eigenstates (= eigenvectors) ϕ_i of the generator matrix followed by taking the absolute square: $P(j_i) = |\phi_i^\dagger \psi|^2$, where $J\phi_i = j_i \phi_i$ and $i = 1, 2, \ldots$ labels the eigenstates ϕ_i and eigenvalues j_i of the generator (in Dirac notation: $P(m) = |\langle m | \psi \rangle|^2$, where $J|m\rangle = m|m\rangle$ and the eigenvalues m label the corresponding eigenstates $|m\rangle$). The observed spin value is always one of the eigenvalues j_i and occurs with probability $P(j_i)$. In particular, the eigenstate $\psi = \phi_i$ has the *definite* spin value j_i, that is, $P(j_i) = 1$. The total spin, as opposed to the spin component, determines what representation the generator matrix needs to be taken from (e.g., for spin ½ we use the 2-dimensional representation, for spin 1 we use the 3-dimensional representation, etc.)

Note that when generators are interpreted as quantum observables, they must be appropriately normalized: The spin-½ generators must have eigenvalues +½ and −½, the spin-1 generators must have eigenvalues +1, 0, −1, etc. The basis generators shown in the diagram are normalized in this way.

We know that no pair of the three basis generators J_x, J_y, and J_z commutes: $[J_i, J_j] \neq 0$ for $i \neq j$. Consequently, two basis generators cannot have (all) the eigenvectors in common. This leads to the hard-to-swallow fact that only *one* component of the spin can have a *definite* value! This is a form of Heisenberg's *Uncertainty Principle*. While all three spin components have expectation values, only one of them can have a definite value. It is conventional to choose J_z for the spin component with the definite value. (We can prepare a spin with a definite J_z value by measuring it along the z axis.)

It is a general feature of quantum mechanics that observables, such as spin, momentum, energy, etc., can be identified with generators (= Lie-algebra elements). Therefore, every observable is automatically associated with a set of transformations (= Lie-group elements). In our example, spin is associated with SU(2) transformations, which correspond to 3D rotations. What is the significance of these transformations? If the transformations associated with a particular observable leave the law of time evolution unchanged (= symmetry transformations), then the observable is time independent (= conserved)! For example, full 3D rotational symmetry implies the conservation of angular momentum along all axes. Specifically, all three expectation values $\langle J_x \rangle = \psi(t)^\dagger J_x \psi(t)$, $\langle J_y \rangle = \psi(t)^\dagger J_y \psi(t)$, and $\langle J_z \rangle = \psi(t)^\dagger J_z \psi(t)$ will be conserved. (See the Appendix "Symmetry and Conservation in Quantum Mechanics" for more on this topic.)