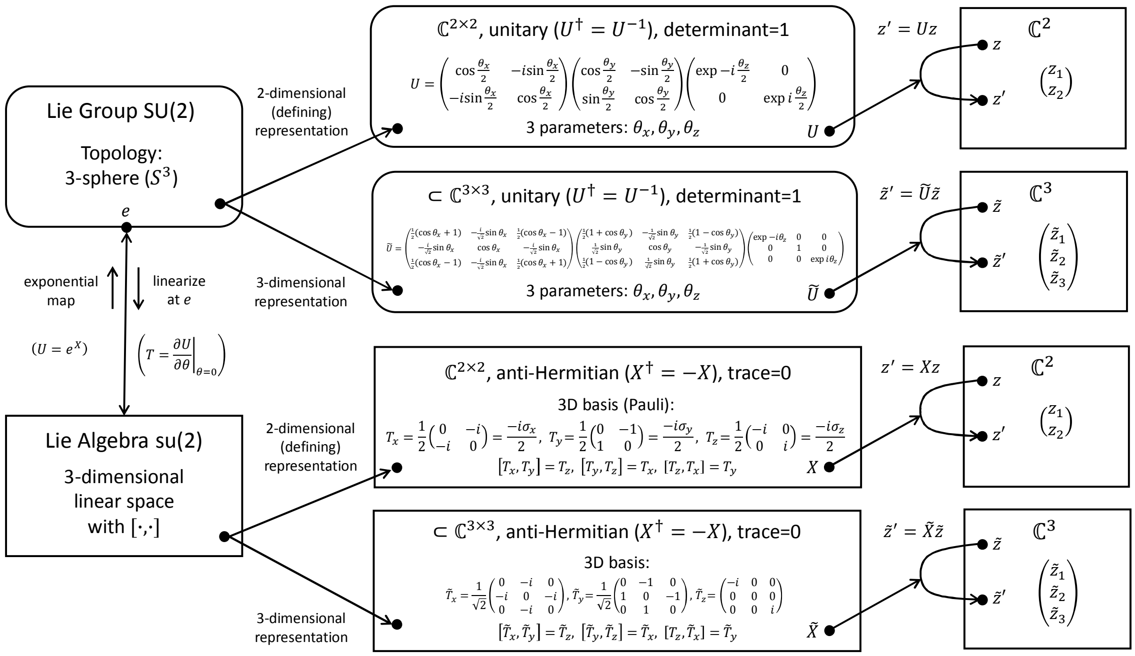
3. Rotation in Our Three-Dimensional Space and Angular Momentum

3.1 SU(2): The Special Unitary Group of Degree Two



The Lie group SU(2) is a good example to start with. It is rich enough to illustrate most features of Lie groups, yet simple enough to serve as a first example. Moreover, it is one of the most important groups in physics! The group, or rather its *defining representation*, consists of all 2×2 unitary matrices U with determinant one. The S in SU(2) stands for *special*, indicating that $\det(U) = U_{11}U_{22} - U_{12}U_{21} = 1$ and the U stands for *unitary*, meaning that the complex matrix U satisfies $U^\dagger U = I$, where the dagger † indicates the Hermitian conjugate (= transpose and complex conjugate) and I is the 2×2 identity matrix.

How many parameters do we need to describe a general SU(2) matrix? For a general complex 2×2 matrix we need four complex or eight real parameters. It turns out that the constraints $U^\dagger U = I$ and $\det(U) = 1$ reduce the number of free parameters from eight to three. In other words, the group SU(2) is a 3-dimensional manifold. As we will see later, it has the shape of a 3-sphere.

There are many ways of parametrizing an SU(2) matrix. One possibility is shown in the diagram (upper branch). The matrix $U(\theta_x, \theta_y, \theta_z)$ is written as a product of three matrices, where each one depends on only a single parameter: $U_x(\theta_x) \cdot U_y(\theta_y) \cdot U_z(\theta_z)$. If all three parameters are set to zero, the overall matrix becomes the identity matrix: $U(0,0,0) = I$. The transformation matrix U acts on complex 2-component vectors like $z' = Uz$, where $z, z' \in \mathbb{C}^2$ are elements of the representation space.

The Lie algebra that goes with the Lie group SU(2) is denoted su(2). To find the k-th basis generator, we take the derivative of the transformation matrix $U(\theta_x, \theta_y, \theta_z)$ with respect to the k-th parameter, θ_k, where $k = x, y, z$, and evaluate the result for $\theta_k = 0$. The resulting three 2×2 matrices T_x, T_y, T_z are shown in the diagram. These matrices are related to the three *Pauli matrices* $\sigma_x, \sigma_y, \sigma_z$ as follows: $T_k = -i\sigma_k/2$ [PfS, Ch. 3.4.3]. A general element X of the Lie algebra is a linear combination of the three basis generators, $X = \alpha T_x + \beta T_y + \gamma T_z$, where α, β, γ are real coefficients.

The Lie bracket of two generators is given by the matrix commutator: $[X, Y] = XY - YX$. Evaluating the commutation relations of the basis generators yields $[T_x, T_y] = T_xT_y - T_yT_x = T_z$, $[T_y, T_z] = T_yT_z - T_zT_y = T_x$, and $[T_z, T_x] = T_zT_x - T_xT_z = T_y$. Note that in this algebra, no pair of (distinct) basis generators commutes: $[T_i, T_j] \neq 0$ for $i \neq j$. The commutator of any two basis generators can always be expressed as a linear combination of the basis generators, for example, $[T_z, T_y] = -1\,T_x + 0\,T_y + 0\,T_z$. All nine of these relations can be written compactly as $[T_i, T_j] = \sum_{k=1}^{3} \varepsilon_{ijk} T_k$, where the T_x, T_y, T_z are now numbered as T_1, T_2, T_3 and ε_{ijk} is the 3-dimensional *Levi-Civita symbol* defined as $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$ and $\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1$, else $\varepsilon_{ijk} = 0$ [PfS, Ch. B.5.5]. The coefficients ε_{ijk} in the above commutation relations are the *structure constants* of su(2).

The matrices X that make up the su(2) algebra are of a particular form that is dictated by the form of the matrices U that make up the SU(2) group. From the fact that the SU(2) matrices are unitary, $U^\dagger U = I$, it can be shown that the su(2) matrices must be anti-Hermitian, $X^\dagger = -X$, and from the fact that the SU(2) matrices have determinant one, $\det(U) = U_{11}U_{22} - U_{12}U_{21} = 1$, it can be shown that the su(2) matrices must have trace zero, $\mathrm{tr}(X) = X_{11} + X_{22} = 0$ (see [GTNut, Ch. IV.4; PfS, Ch. 3.4.3]). The matrices X are elements of a 3-dimensional *real* vector space; only real combinations of the three (complex) basis generators are allowed to ensure that the result remains anti-Hermitian.

To get from the Lie algebra back to the Lie group, we use the exponential map. The matrix exponential $\exp(T_x\theta_x)$ yields the first matrix $U_x(\theta_x)$ shown in the diagram. (The Wolfram Alpha command `matrixexp({{0,-i},{-i,0}}*x/2)` can be used to check this.) Similarly, the basis generators T_y and T_z produce the matrices $U_y(\theta_y)$ and $U_z(\theta_z)$. Putting it all together, we have $U(\theta_x, \theta_y, \theta_z) = \exp(T_x\theta_x) \cdot \exp(T_y\theta_y) \cdot \exp(T_z\theta_z)$. Incidentally, the parameters $\theta_x, \theta_y, \theta_z$ in this exponential expression agree with those in our initial matrix expression.

Now we can see how to construct the transformation matrix U shown in the diagram: First, from the defining conditions of the transformation matrix (Lie-group element) derive the defining conditions of the generator (Lie-algebra element). Then, from the latter conditions find a basis for the generators. Finally, use the exponential map to determine the transformation matrix.

Remarkably, SU(2) has not only the 2-dimensional representation we discussed so far but has also a *3-dimensional* one! The lower branch of the diagram shows the corresponding 3×3 transformation matrix $\widetilde{U}(\theta_x, \theta_y, \theta_z)$. It depends on the same three parameters as the 2-dimensional representation but acts on complex 3-vectors instead of 2-vectors. Taking the derivatives, we find the corresponding three basis generators in the Lie algebra, $\tilde{T}_x, \tilde{T}_y, \tilde{T}_z$, which are also 3×3 matrices. The exponential map takes us back to the group matrices. Evaluating the Lie brackets, we find the exact same commutation relations (and thus the same structure constants) as for the 2-dimensional representation. Both are representations of the same group and the same algebra! It turns out that SU(2) has representations in all dimensions, from one to infinity. Later, we will see how to construct them systematically.

Note that for the defining representation, the transformations $U(0,0,0)$ and $U(0,0,2\pi)$ are *different*, namely I and $-I$, but for the 3-dimensional representation, the transformations $\widetilde{U}(0,0,0)$ and $\widetilde{U}(0,0,2\pi)$ are the *same*, namely I. Since the map from the abstract group to the defining representation is necessarily one-to-one, the map to the 3-dimensional representation must be many-to-one (in this case two-to-one). That's ok, a representation does *not* have to be one-to-one, it only needs to preserve the algebraic structure, that is, it must be a *homomorphism*.