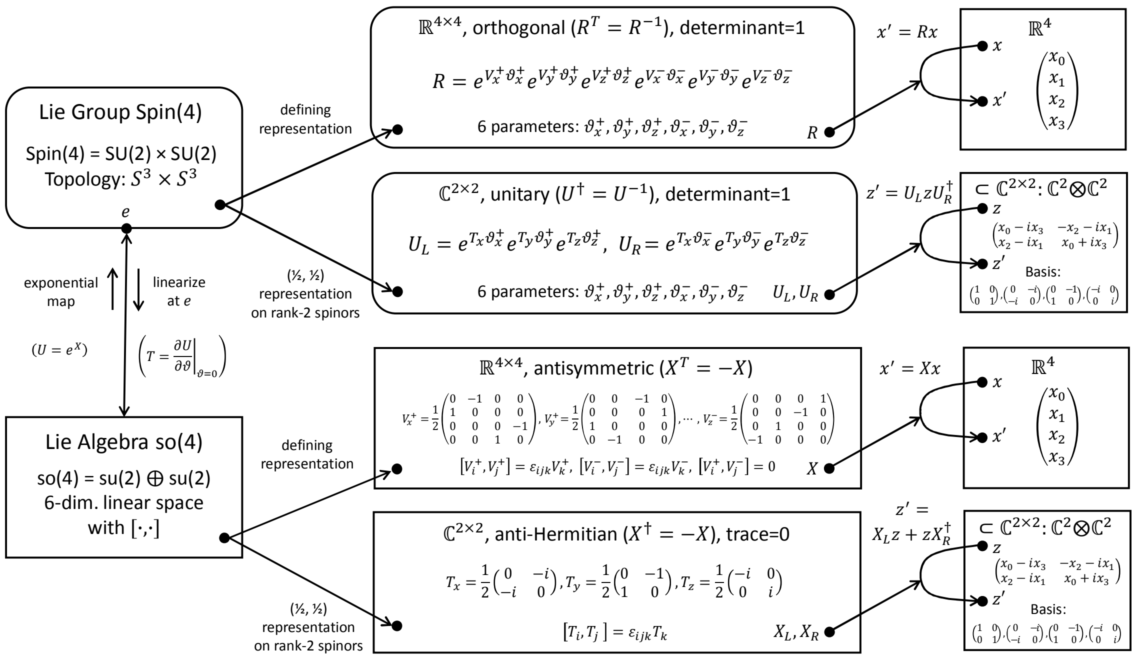
5.9 Spin(4): The (½, ½) Representation; 4D Rotation with Two SU(2) Matrices



Having practiced with the (1, 1) representation, we are now ready to tackle the important (½, ½) representation. We construct it by taking the tensor product (½, 0) ⊗ (0, ½), which acts on complex 2×2 matrices known as *rank-2 spinors* (see the lower branch of the diagram). Amazingly, this representation is equivalent to the (complexified) defining representation of Spin(4), which is shown again for reference in the upper branch of the diagram! At first glance, this seems unbelievable: how can a representation acting on exotic rank-2 spinors be equivalent to one acting on good old-fashioned 4D vectors? At the end of this example, it will make sense!

How does the rank-2 spinor z transform under the (½, ½) representation? Consider the outer product of the spinor z_L from the (½, 0) representation space and the spinor z_R from the (0, ½) representation space: $z = z_L z_R^T$. Knowing that the (½, 0) representation acts like $z_L' = U_L(\vartheta_x^+, \vartheta_y^+, \vartheta_z^+) z_L$ and that the (0, ½) representation acts like $z_R' = U_R(\vartheta_x^-, \vartheta_y^-, \vartheta_z^-) z_R$, we conclude that z transforms like $z' = z_L' z_R'^T = U_L z_L (U_R z_R)^T = U_L z_L z_R^T U_R^T = U_L z U_R^T$. Now, it turns out that to get a direct correspondence between the rank-2 spinor and the vector representations (without the need for a similarity transformation), we need to pick the (equivalent) complex-conjugate representation of (0, ½). Thus, given the U_L and U_R matrices shown in the diagram, the transformation is $z' = U_L z U_R^\dagger$. In conclusion, each element of Spin(4) maps to two SU(2) matrices, U_L and U_R, which act jointly on the rank-2 spinor z and collectively depend on six parameters (see the lower branch of the diagram).

To find the six basis generators and their action on the rank-2 spinor z, we take the derivative of the transformation $z' = U_L z U_R^\dagger$ with respect to the six parameters and evaluate the result at $\vartheta_i^+ = \vartheta_i^- = 0$. For the three parameters ϑ_i^+ we find $z' = T_i z$ and for the three parameters ϑ_i^- we find $z' = z T_i^\dagger$, where $T_i = -i\sigma_i/2$. Thus, the action of an arbitrary generator is $z' = X_L z + z X_R^\dagger$, where X_L and X_R are two 2×2 matrices in the basis T_x, T_y, T_z. In conclusion, each element of so(4) maps to two (anti-Hermitian) 2×2 matrices, X_L and X_R, which act jointly on the rank-2 spinor z (see the lower branch of the diagram).

The (½, ½) representation acts on complex 2×2 matrices and thus is *four* dimensional if the matrices are considered elements of a *complex vector space* (in which we use *complex* numbers to combine vectors). However, if the matrices are considered elements of a *real vector space* (in which we use only *real* numbers to combine vectors), the same representation is *eight* dimensional. In the latter case, the 8-dimensional representation is reducible into two 4-dimensional ones. Specifically, the decomposition of a general complex 2×2 matrix

$$z + \tilde{z} = \begin{pmatrix} x_0 - ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & x_0 + ix_3 \end{pmatrix} + i \begin{pmatrix} x_4 - ix_7 & -x_6 - ix_5 \\ x_6 - ix_5 & x_4 + ix_7 \end{pmatrix},$$

where the x_i are *real* numbers, has the property that the transformation $z' = U_L z U_R^\dagger$ preserves the form of each part. A matrix of the form z (the first part) lives in the 4-dimensional subspace spanned by the basis I, $-i\sigma_x$, $-i\sigma_y$, $-i\sigma_z$. Why is this subspace invariant under $z' = U_L z U_R^\dagger$? Because multiplying three unitary matrices yields again a unitary matrix (remember, they form a group): the matrices U_L and $U_R^\dagger$ are unitary by definition and the normalized $z/\sqrt{\det z}$ is unitary as well (note that $zz^\dagger = \det z$) [GTNut, Ch. IV.7, p. 274]. A similar argument can be made for a matrix of the form $\tilde{z}$ (the second part), which lives in the 4-dimensional subspace spanned by the basis iI, σ_x, σ_y, σ_z.

Now, the claim is that the (½, ½) representation of Spin(4) acting on the 4-dimensional subspace defined by z (or $\tilde{z}$) is equivalent to the defining representation. To check this with an example, we pick the self-dual double rotation for which $\vartheta_z^+ \neq 0$ and $\vartheta_x^+ = \vartheta_y^+ = \vartheta_x^- = \vartheta_y^- = \vartheta_z^- = 0$. In this case, the SU(2) matrix pair is $U_L = \exp(T_z \vartheta_z^+)$ and $U_R = 1$ and the rank-2 spinor transforms like $z' = \exp(T_z \vartheta_z^+)\, z$:

$$\begin{pmatrix} x_0' - ix_3' & -x_2' - ix_1' \\ x_2' - ix_1' & x_0' + ix_3' \end{pmatrix} = \begin{pmatrix} e^{-i\vartheta_z^+/2} & 0 \\ 0 & e^{i\vartheta_z^+/2} \end{pmatrix} \begin{pmatrix} x_0 - ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & x_0 + ix_3 \end{pmatrix}.$$

Multiplying out the matrices and solving for the x_i', we find that the vector x_i transforms like

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (e^{-i\vartheta_z^+/2} + e^{i\vartheta_z^+/2}) & 0 & 0 & -i(e^{-i\vartheta_z^+/2} - e^{i\vartheta_z^+/2}) \\ 0 & (e^{-i\vartheta_z^+/2} + e^{i\vartheta_z^+/2}) & -i(e^{-i\vartheta_z^+/2} - e^{i\vartheta_z^+/2}) & 0 \\ 0 & i(e^{-i\vartheta_z^+/2} - e^{i\vartheta_z^+/2}) & (e^{-i\vartheta_z^+/2} + e^{i\vartheta_z^+/2}) & 0 \\ i(e^{-i\vartheta_z^+/2} - e^{i\vartheta_z^+/2}) & 0 & 0 & (e^{-i\vartheta_z^+/2} + e^{i\vartheta_z^+/2}) \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Magically, all the matrix components are real sines and cosines ($\sin\theta = i[\exp(-i\theta) - \exp(i\theta)]/2$ and $\cos\theta = [\exp(-i\theta) + \exp(i\theta)]/2$), and the transformation matrix exactly matches the self-dual double-rotation matrix $R_z(\vartheta_z^+)$ from our earlier example! Repeating this exercise for the remaining five double rotations reveals that the two representations are, in fact, equivalent.

Just to be sure, let's also check the action of an algebra element. The generator pair for the above double rotation is $X_L = T_z$ and $X_R = 0$, that is, the rank-2 spinor transforms like $z' = T_z z$:

$$\begin{pmatrix} x_0' - ix_3' & -x_2' - ix_1' \\ x_2' - ix_1' & x_0' + ix_3' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} x_0 - ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & x_0 + ix_3 \end{pmatrix}.$$

Solving for the x_i', indeed recovers the basis generator V_z^+ from our earlier example:

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$