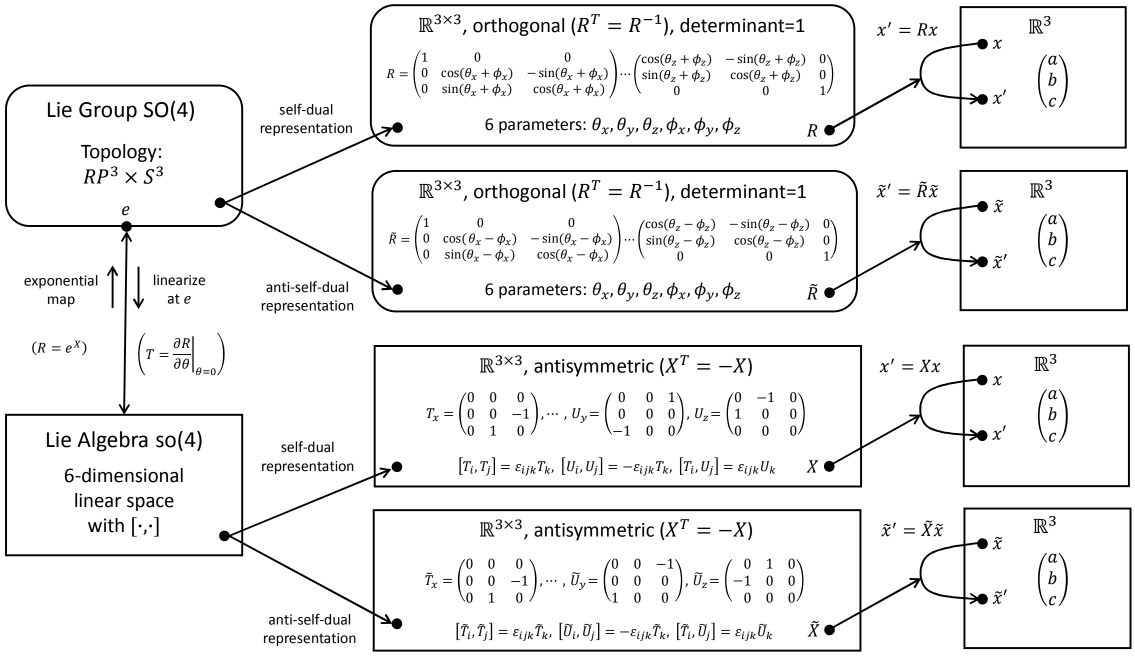
5.4 SO(4): Self-Dual and Anti-Self-Dual Representations on 3D Vectors



In the previous example, we found two 3-dimensional representations of SO(4), one acting on self-dual and one acting on anti-self-dual tensors. It is instructive to rewrite these representations in a form where they act on 3-component column vectors. For example, letting the matrix for 4D rotation in the yz plane (from the previous example) act on a general self-dual matrix, $Y^+ \to RY^+R^T$, yields

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{pmatrix} \to \begin{pmatrix} 0 & a & b\cos\theta_x - c\sin\theta_x & b\sin\theta_x + c\cos\theta_x \\ -a & 0 & b\sin\theta_x + c\cos\theta_x & -(b\cos\theta_x - c\sin\theta_x) \\ -(b\cos\theta_x - c\sin\theta_x) & -(b\sin\theta_x + c\cos\theta_x) & 0 & a \\ -(b\sin\theta_x + c\cos\theta_x) & b\cos\theta_x - c\sin\theta_x & -a & 0 \end{pmatrix},$$

which is again a self-dual matrix, as expected. Now, focusing on the three free parameters a, b, and c, the same transformation can be written in the simpler (unpacked) form

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} a \\ b\cos\theta_x - c\sin\theta_x \\ b\sin\theta_x + c\cos\theta_x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = R_{yz}\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Repeating this procedure for the remaining five factors of R yields the overall 3×3 transformation matrix $R' = R_{yz}(\theta_x) \cdot R_{zx}(\theta_y) \cdot R_{xy}(\theta_z) \cdot R_{wx}(\phi_x) \cdot R_{wy}(\phi_y) \cdot R_{wz}(\phi_z)$, where

$$R_{yz} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{pmatrix},\ R_{zx} = \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix},\ R_{xy} = \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_{wx} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_x & -\sin\phi_x \\ 0 & \sin\phi_x & \cos\phi_x \end{pmatrix},\ R_{wy} = \begin{pmatrix} \cos\phi_y & 0 & \sin\phi_y \\ 0 & 1 & 0 \\ -\sin\phi_y & 0 & \cos\phi_y \end{pmatrix},\ R_{wz} = \begin{pmatrix} \cos\phi_z & -\sin\phi_z & 0 \\ \sin\phi_z & \cos\phi_z & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Interestingly, these are just ordinary 3D rotations about the three coordinate axes of the representation space! Moreover, rotating by the angle θ_x has the same effect as rotating by the angle ϕ_x, etc.

We can simplify the expression for the overall transformation by combining rotations by θ_i and ϕ_i for the same i into the same matrix. This reduces the number of factors from six to three (see the upper branch of the diagram):

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x + \phi_x) & -\sin(\theta_x + \phi_x) \\ 0 & \sin(\theta_x + \phi_x) & \cos(\theta_x + \phi_x) \end{pmatrix} \begin{pmatrix} \cos(\theta_y + \phi_y) & 0 & \sin(\theta_y + \phi_y) \\ 0 & 1 & 0 \\ -\sin(\theta_y + \phi_y) & 0 & \cos(\theta_y + \phi_y) \end{pmatrix} \begin{pmatrix} \cos(\theta_z + \phi_z) & -\sin(\theta_z + \phi_z) & 0 \\ \sin(\theta_z + \phi_z) & \cos(\theta_z + \phi_z) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that although this simplified matrix is different from the original one because we rearranged the factors for the combining, it is the same representation, just parametrized in a different way.

Similar to what we did for the transformation matrices, we can also unpack the basis generators. In fact, we already did this for T_x in the previous example:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} 0 \\ -c \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = T_x \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Repeating this procedure for the remaining five basis generators, we find that the three T_i are just the generators we had earlier for so(3) and $U_i = T_i$. Alternatively, we could have taken the derivatives of R with respect to its six parameters and evaluated the results at the identity.

Now, let's turn to the anti-self-dual representation $\tilde{R}$. Following the same unpacking procedure as before, we find that the first three matrices are the same as in the self-dual case, $\tilde{R}_{yz} = R_{yz}$, $\tilde{R}_{zx} = R_{zx}$, and $\tilde{R}_{xy} = R_{xy}$, but the second three are inverted: $\tilde{R}_{wx} = R_{wx}^{-1}$, $\tilde{R}_{wy} = R_{wy}^{-1}$, and $\tilde{R}_{wz} = R_{wz}^{-1}$. Again, we can simplify the overall transformation by combining matrices that rotate in the same plane:

$$\tilde{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x - \phi_x) & -\sin(\theta_x - \phi_x) \\ 0 & \sin(\theta_x - \phi_x) & \cos(\theta_x - \phi_x) \end{pmatrix} \begin{pmatrix} \cos(\theta_y - \phi_y) & 0 & \sin(\theta_y - \phi_y) \\ 0 & 1 & 0 \\ -\sin(\theta_y - \phi_y) & 0 & \cos(\theta_y - \phi_y) \end{pmatrix} \begin{pmatrix} \cos(\theta_z - \phi_z) & -\sin(\theta_z - \phi_z) & 0 \\ \sin(\theta_z - \phi_z) & \cos(\theta_z - \phi_z) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The first three basis generators are the same as in the self-dual case, $\tilde{T}_i = T_i$, but the second three have the opposite sign: $\tilde{U}_i = -U_i$ (see the lower branch of the diagram).

We found that the 3-dimensional self-dual and anti-self-dual representations of SO(4) consist of ordinary 3D rotations. Are the self-dual and anti-self-dual representations equivalent? No, despite looking very similar, they are not! They cannot be turned into each other with a similarity transformation (= change of basis of the representation space), that is, $\tilde{R} \neq SRS^{-1}$ for any S. The reason for this is that the dependence of the two matrices on the six parameters is rather different: The first one depends on the sum of two plane-rotation angles and the second one on the difference of two plane-rotation angles. In a previous example, we labeled these two representations as $\mathbf{3}$ and $\overline{\mathbf{3}}$.

This example suggests that it may be advantageous to replace the six rotation parameters θ_i and ϕ_i by the sum $\vartheta_i^+ = \theta_i + \phi_i$ and difference $\vartheta_i^- = \theta_i - \phi_i$. We'll call the original parameters *plane-rotation* parameters and the new ones *self-dual* and *anti-self-dual double-rotation* parameters. In the next example, we'll carry out this parameter change for the defining representation of SO(4).