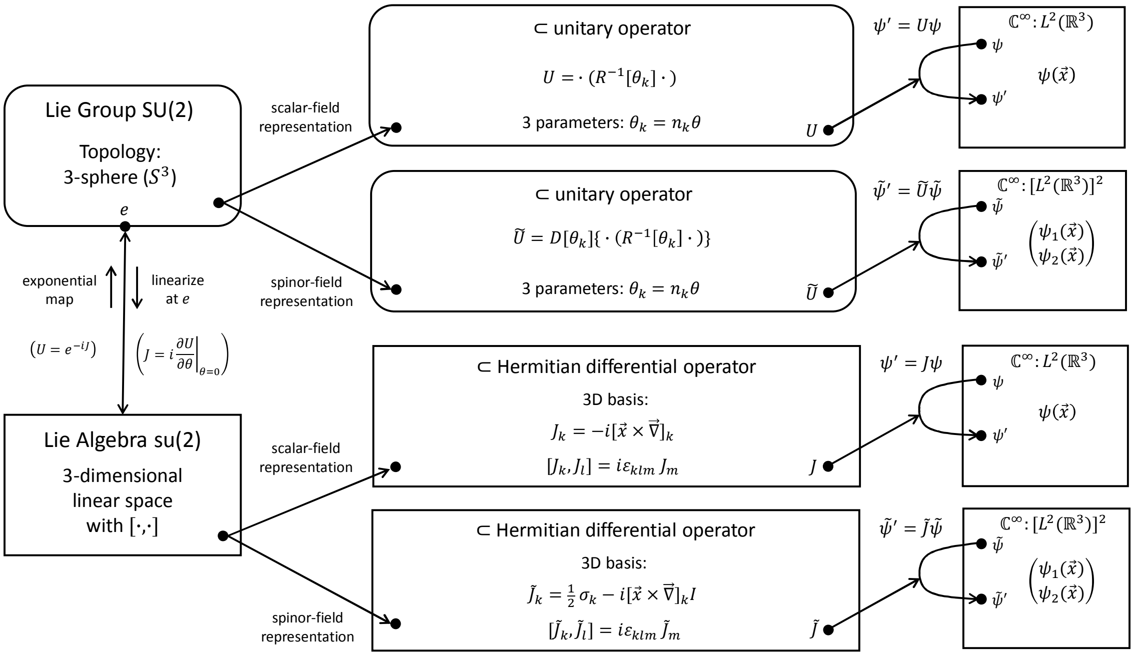
3.14 SU(2): Spinor-Field Representation; Two Types of Angular Momenta



First, we discussed the *spin* of a particle at a fixed location, then, we discussed the *orbital angular momentum* of a non-spinning (scalar) particle roaming through space. Now, it's time to tackle a particle with spin ½ roaming through space! To describe this kind of particle, an electron for example, we need a wave function with two components: an amplitude for "spin up" and an amplitude for "spin down" at every point in space: $(\psi_{\text{up}}(\vec{x}), \psi_{\text{down}}(\vec{x}))^T$. In other words, we need a *spinor field*. How does such a field transforms under SU(2) and what are the generators of this representation? The answer to the latter question will provide us with the operators for the angular momentum of a spinor field.

The upper branch of the diagram shows again the representation acting on a scalar field, but this time written in a more compact notation. We packed the three coordinates into the vector $\vec{x} = (x, y, z)^T$ and the three partial derivatives into the vector $\vec{\nabla} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)^T$. Moreover, we used the axis-angle parameters $\theta_k = n_k\theta$, where n_k is the unit vector along the axis, $k = 1, 2, 3$, and θ is the rotation angle about this axis. Finally, we used the Einstein summation convention, which implies a summation over products with repeated indices. A general element of the Lie algebra can now be written as $J = J_k\theta_k$ (summation implied), where $J_k = -i[\vec{x}\times\vec{\nabla}]_k$ are the (Cartesian) basis generators (expressed as the k-th component of the cross product). Exponentiating J yields the Lie-group element $U = e^{-iJ_k\theta_k}$ (summation in the exponent implied). This operator is the same thing that we earlier wrote as $U = \cdot\,(R^{-1}[\theta_k]\,\cdot)$. Thus, we can write SU(2) transformations of a scalar field in two equivalent ways: $\psi'(\vec{x}) = e^{-iJ_k\theta_k}\psi(\vec{x})$ or $\psi'(\vec{x}) = \psi(R^{-1}[\theta_k]\,\vec{x})$.

Moving on to the spinor field $\tilde{\psi}(\vec{x}) = (\psi_1(\vec{x}), \psi_2(\vec{x}))^T$, an SU(2) transformation now must do two things: first it must rotate the function in space (as in the scalar case) and second it must rotate the spinor at each point in space. Phrased differently, the transformation now consists of two parts, a first part that acts on the coordinates of the function's argument and a second part that mixes the components of the function [PfS, Ch. 3.7.11]. For the first part we use again $R^{-1}[\theta_k]$ acting on the position vector $\vec{x}$ and for the second part we use the 2×2 matrix $D[\theta_k]$ acting on the spinor $\tilde{\psi}$, where $D[\theta_k]$ is the 2-dimensional (defining) representation of SU(2), which we discussed earlier but now write as D instead of U. Thus, the overall transformation is $\tilde{\psi}'(\vec{x}) = D[\theta_k]\,\tilde{\psi}(R^{-1}[\theta_k]\,\vec{x})$. Splitting off the transformation operator from the spinor field using our informal dot notation, we get $\tilde{U} = D[\theta_k]\{\cdot(R^{-1}[\theta_k]\,\cdot)\}$, where, as before, we have to insert the field's name at the first dot and the position argument at the second dot (see the diagram).

What are the basis generators of this spinor-field representation? First, we know that $D[\theta_k] = e^{-iS_k\theta_k}$, where $S_k = \sigma_k/2$ are the basis generators of the 2-dimensional representation. Then, we can write $\{\cdot(R^{-1}[\theta_k]\,\cdot)\} = e^{-iL_k\theta_k}$, where $L_k = -i[\vec{x}\times\vec{\nabla}]_k I$ are the basis generators of the infinite-dimensional representation. The L_k are almost the same as the J_k from the scalar case, except that we have to multiply them with the 2×2 identity matrix I so that they can act on 2-component spinors. Combining the two parts, we get the overall transformation $\tilde{U} = e^{-iS_k\theta_k}e^{-iL_k\theta_k}$. Using the fact that the generators S_k and L_k commute, we can write $\tilde{U} = e^{-i(S_k+L_k)\theta_k} = e^{-i\tilde{J}_k\theta_k}$, where $\tilde{J}_k = S_k + L_k = \sigma_k/2 - i[\vec{x}\times\vec{\nabla}]_k I$ are the three basis generators of the spinor-field representation (see the diagram). In quantum mechanics, $\tilde{J}_z$ is the operator for the angular momentum along the z axis. For the spinor field, this angular momentum is composed of spin *and* orbital angular momentum and we call it the *overall* angular momentum (*total* angular momentum is already used for $\tilde{J}_x^2 + \tilde{J}_y^2 + \tilde{J}_z^2$).

For example, the generator of rotation about the z axis is $\tilde{J}_z = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - i(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ or, after switching to spherical coordinates, $\tilde{J}_z = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - i\frac{\partial}{\partial\phi}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Finally, letting this generator act on the arbitrary spinor field $\tilde{\psi} = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\sin\Theta\, e^{i\phi}\\ \cos\Theta\end{pmatrix}$, we get $\tilde{\psi}' = \tilde{J}_z\tilde{\psi} = \frac{1}{2}\begin{pmatrix}\sin\Theta\, e^{i\phi}\\ -\cos\Theta\end{pmatrix} + \begin{pmatrix}\sin\Theta\, e^{i\phi}\\ 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3\sin\Theta\, e^{i\phi}\\ -\cos\Theta\end{pmatrix}$.

The arbitrary spinor field from above is *not* an eigenfunction of the generator $\tilde{J}_z$. But we can easily construct eigenfunctions of $\tilde{J}_z = S_z + L_z$ by multiplying an eigenvector of S_z with an eigenfunction of L_z. The corresponding eigenvalues are just the eigenvalues of S_z plus the eigenvalues of L_z. For example, combining the spin eigenvector $(1,0)^T$ of S_z with eigenvalue ½ and the orbital eigenfunction $\sin\Theta\, e^{-i\phi}$ of L_z with eigenvalue −1 yields the spinor field $\tilde{\psi} = \begin{pmatrix}1\\ 0\end{pmatrix}\sin\Theta\, e^{-i\phi}$. This spinor field satisfies the eigenequation $\tilde{J}_z\tilde{\psi} = (\frac{1}{2} - 1)\tilde{\psi}$ and thus has an overall angular momentum of −½ (along the z axis).

To summarize: There are two types of angular momenta because angular momentum is the generator of rotation and we need to rotate (i) the field's location in space and (ii) the orientation of the field's components! Although orbital angular momentum and spin seem qualitatively rather different, their operators can simply be added together. Furthermore, for states of definite angular momentum (= eigenstates), their values (= eigenvalues of the states) can simply be added together as well.

It is easy to generalize this spinor-field example to other types of fields, such as a vector field. We simply replace the 2-dimensional representation of SU(2) with, say, the 3-dimensional representation for a vector field. In other words, we only need to update the transformation matrix $D[\theta_k]$ and the associated generators S_k, everything else remains the same.