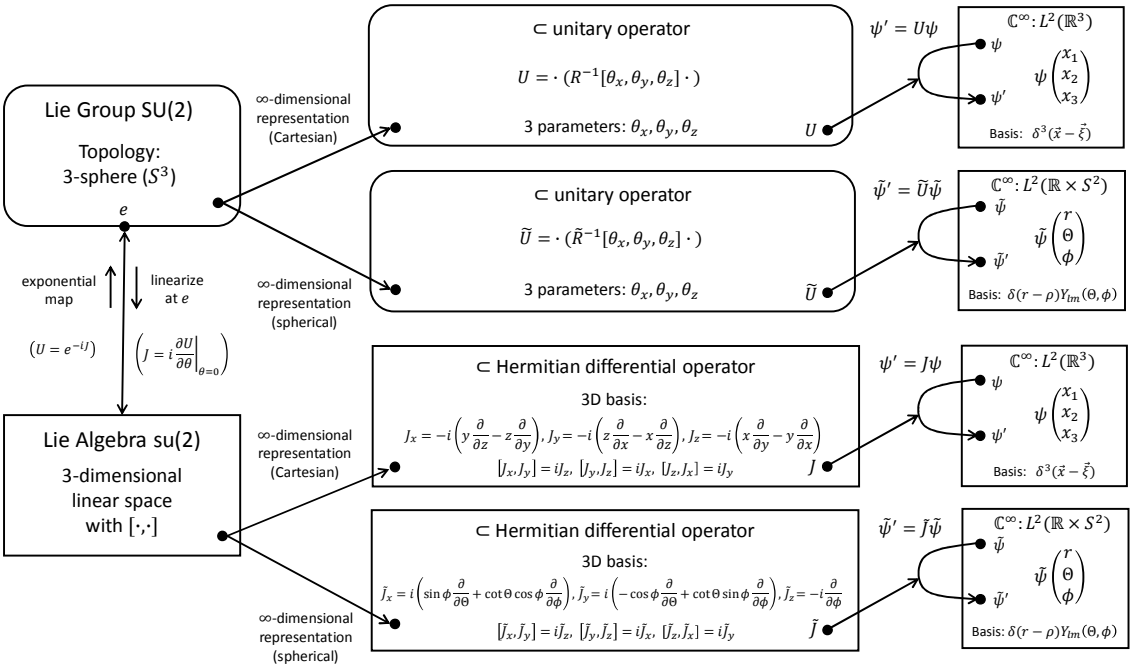
3.12 SU(2): Application to Orbital Angular Momentum; Spherical Harmonics



In the previous example, we introduced an infinite-dimensional representation of SU(2) that acts on the function $\psi(\vec{x})$, where the components of $\vec{x} = (x_1, x_2, x_3)^T$ are Cartesian (position) coordinates (see the upper branch of the diagram). However, for problems with spherical symmetry, it is more natural to write this function as $\tilde{\psi}(\vec{s})$, where the components of $\vec{s} = (r, \Theta, \phi)^T$ are now spherical (position) coordinates: r is the radius, Θ is the colatitude, and ϕ is the longitude (such that $x_1 = r\sin\Theta\cos\phi$, $x_2 = r\sin\Theta\sin\phi$, and $x_3 = r\cos\Theta$). When rewritten in spherical coordinates, the three basis generators J_i become the $\tilde{J}_i$ shown in the lower branch of the diagram [QTGR, Ch. 8.3].

A simple basis for the functions in the representation space is the position basis. For Cartesian coordinates, this basis consists of the functions $\delta^3(\vec{x} - \vec{\xi})$, an infinitude of Dirac pulses, one at every possible position $\vec{\xi} = (\xi_1, \xi_2, \xi_3)^T$. An arbitrary function can now be written as a weighted sum (actually, integral) of these pulses (see the upper branch of the diagram). However, we know that a more meaningful basis is given by the *eigenfunctions* of the generators. In quantum mechanics, these basis functions are the wave functions for which the respective observable (here: angular momentum) has a *definite* value. Working in spherical coordinates, we pick the basis generator $\tilde{J}_z = -i\partial/\partial\phi$ and solve the eigenequation $\tilde{J}_z\Psi_m(\vec{s}) = m\Psi_m(\vec{s})$. Keeping in mind that the function $\tilde{\psi}(\vec{s})$, and thus the eigenfunctions $\Psi_m(\vec{s})$, must assume the same value for ϕ and $\phi + 2\pi$, we find the solutions $\Psi_m(r, \Theta, \phi) = F(r, \Theta)\cdot e^{im\phi}$, where $F(r, \Theta)$ is an arbitrary function of r and Θ and the associated eigenvalues are $m = 0, \pm1, \pm2$, etc. So, we get nice orthogonal basis functions of ϕ (e.g., along the equator of the unit sphere), but we do *not* get unique basis functions on the whole 3-dimensional (r, Θ, ϕ)-space because any function $F(r, \Theta)$ satisfies the above eigenequation.

What can we do to get unique orthogonal functions on the whole space? The trick is to use a second operator and pick the functions that are eigenfunctions of *both* operators. For the two operators to have joint eigenfunction, they must commute. So, operators $\tilde{J}_x$ and $\tilde{J}_y$ are no good. However, the *Casimir*

operator $\tilde{J}_C^2 = \tilde{J}_x^2 + \tilde{J}_y^2 + \tilde{J}_z^2$, which we met earlier, commutes with $\tilde{J}_z$ as well as the other basis generators. In spherical coordinates, the Casimir operator evaluates to $\tilde{J}_C^2 = -(1/\sin\Theta)(\partial/\partial\Theta)(\sin\Theta)\partial/\partial\Theta - (1/\sin^2\Theta)\partial^2/\partial\phi^2$ and is independent of r [QTGR, Ch. 8.4]. The joint eigenfunctions can be written as the product $\Psi_{j,m}(r,\Theta,\phi) = F(r)\cdot Y_{j,m}(\Theta,\phi)$, where the $Y_{j,m}(\Theta,\phi)$ are the so-called *spherical harmonics* and $F(r)$ is an arbitrary function of r. The spherical harmonics simultaneously satisfy the two eigenequations $\tilde{J}_C^2 Y_{j,m} = j(j+1)Y_{j,m}$ and $\tilde{J}_z Y_{j,m} = mY_{j,m}$. The eigenvalues $j(j+1)$ and m are given by $j = 0, 1, 2, \ldots$ and $m = 0, \pm1, \pm2, \ldots \pm j$. In general, the functions $Y_{j,m}(\Theta,\phi)$ are quite complicated, but for $j = 0$ and 1 we have: $Y_{0,0} =$ const. (shaped like a sphere), $Y_{1,0} \propto \cos\Theta$ (shaped like a dumbbell aligned with the z axis), $Y_{1,-1} \propto \frac{1}{\sqrt{2}}\sin\Theta\, e^{-i\phi}$, and $Y_{1,+1} \propto -\frac{1}{\sqrt{2}}\sin\Theta\, e^{i\phi}$ [QTGR, Ch. 8.3]. Note that these functions depend on ϕ like $e^{im\phi}$, as we found earlier. Incidentally, the linear combinations $\frac{1}{\sqrt{2}}(Y_{1,-1} - Y_{1,+1}) = \sin\Theta\cos\phi$ and $\frac{1}{\sqrt{2}}\,i(Y_{1,-1} + Y_{1,+1}) = \sin\Theta\sin\phi$ are real-valued functions (shaped like dumbbells aligned with the x and y axes, respectively). To better visualize these functions, search the Internet for pictures and animations of spherical harmonics.

Now, we have nice orthogonal basis functions of ϕ and Θ, but the radial direction is still "unstructured". To complete the orthogonal 3D basis, we must include one more commuting operator. A simple choice is r (the operator that multiplies its target by the radial coordinate), which has the eigenfunctions $\delta(r-\rho)$ (a Dirac pulse at every possible radial coordinate ρ) and the eigenvalues $\rho \in \mathbb{R}$. Thus, a complete set of basis functions is $\Psi_{\rho,j,m}(r,\Theta,\phi) = \delta(r-\rho)\cdot Y_{j,m}(\Theta,\phi)$ labeled by two discrete parameters, j and m, and one continuous one, ρ (see the lower branch of the diagram).

In summary, for our infinite-dimensional representation, $\tilde{J}_z$ alone cannot provide a unique set of orthogonal basis functions. We need three (commuting) operators to achieve this feat. Correspondingly, we need three eigenvalues to label the basis functions: $\Psi_{\rho,j,m}(r,\Theta,\phi)$.

In quantum mechanics, the operator $\tilde{J}_z$ is the observable for the z component of the angular momentum and the Casimir operator $\tilde{J}_C^2$ is the observable for the (squared) total angular momentum. The angular momentum of the spread-out wave function $\tilde{\psi}(r,\Theta,\phi)$ is called *orbital* angular momentum. (In contrast, the "nailed down" particles we discussed earlier had *intrinsic* angular momentum or spin.) The basis functions $\Psi_{\rho,j,m}(r,\Theta,\phi)$ are the wave functions with definite orbital angular momentum: for units in which $\hbar = 1$, their (squared) total magnitude is $j(j+1)$ and their z component is m (in Dirac notation these basis functions would be written as $\langle r,\Theta,\phi|\rho,j,m\rangle$). An important difference between orbital angular momentum and spin is that the former takes on only integer values, $j = 0, 1, 2, \ldots$, whereas the latter can also take on half-integer values $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

In classical mechanics, angular momentum is associated with speed of rotation. But our wave functions do not move; our description does not even include time! As we have discussed earlier, angular momentum in quantum mechanics can be understood as "angular waviness". One wave period wrapping around in a circle implies an angular momentum of one, two wave periods imply an angular momentum of two, etc. That's how electrons in an atom can "orbit" the nucleus (= have orbital angular momentum) without moving (= with a stationary wave function), thus avoiding energy loss due to electromagnetic radiation! Does a wave function ever move? A state of definite energy E contains the time-dependent phase factor e^{-iEt}, but because $\left|e^{-iEt}\right| = 1$ the probability distribution $|\psi|^2$ does not move. However, a *superposition* of states with *different energies* results in a beating among the phase factors, producing a time dependent $|\psi|^2$. (See [FLP, Vol. III, Ch. 7-2] for more details.)