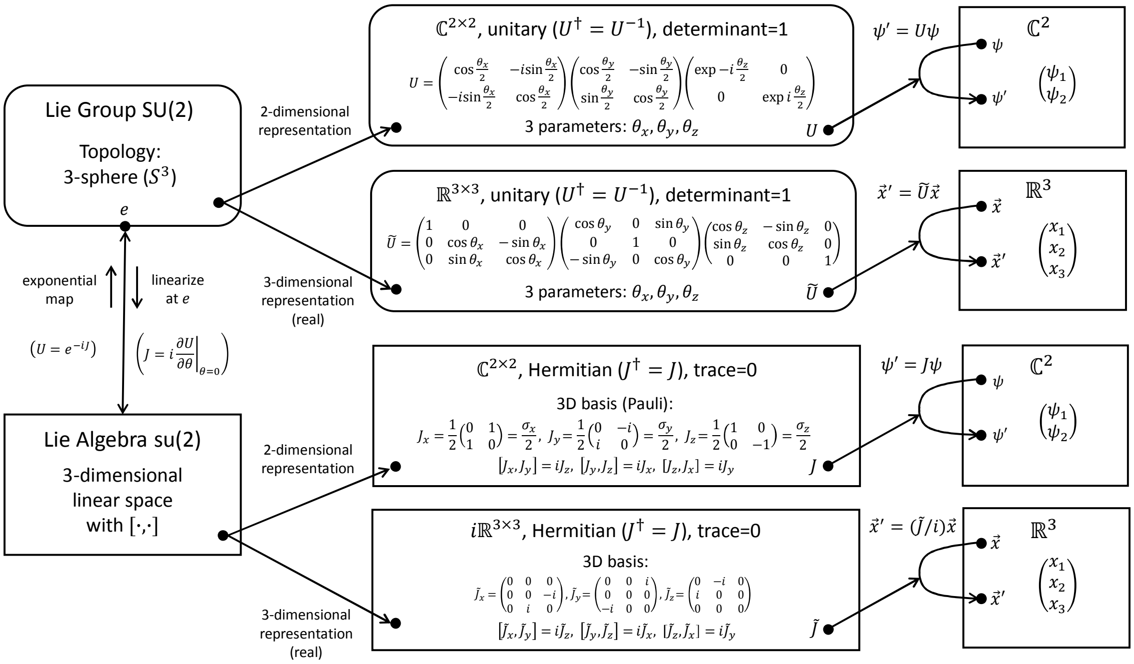
3.9 SU(2): Representation on Real 3-Dimensional Vectors; Spinors vs. Vectors



In the previous example, we saw how the process of "unpacking" the adjoint representation of SU(2) yields a 3-dimensional representation on column vectors. This representation is shown in the lower branch of the diagram and the defining representation is shown again in the upper branch for reference. Note that the new transformation matrix $\tilde{U}$ is real, which means that real vectors $\vec{x}$ always get mapped to real vectors $\vec{x}'$. Given $\vec{x} = (x_1, x_2, x_3)^T = (x, y, z)^T$, we can interpret $\tilde{U}$ as follows: The third factor of the matrix rotates the x, y coordinates by the angle θ_z about the z axis (we derived this part in the previous example), then, the second factor rotates the x, z coordinates about the y axis, and lastly, the first factor rotates the y, z coordinates about the x axis. Unlike our earlier 3-dimensional representation, which rotated complex spin-1 states, this one rotates good old-fashioned Cartesian 3D vectors! (If we let $\tilde{U}$ act on complex vectors, their real and imaginary parts get rotated independently and identically.)

The corresponding basis generators can be found by differentiating the transformation matrix $\tilde{U}$ with respect to its parameters. But we already know from the previous example that the basis generators of the adjoint representation are given by the structure constants as $[\tilde{J}_i]_{kj} = c_{ijk}$, which for su(2) are $c_{ijk} = i\varepsilon_{ijk}$. Either way, we find the basis generators shown in the diagram. They are again traceless Hermitian matrices that satisfy the commutation relations of su(2) and have eigenvalues $-1, 0, 1$. If we had not multiplied the generators by i to make them Hermitian, they would be real like the transformation matrix $\tilde{U}$.

How is this "new" 3-dimensional representation related to the complex spin-1 representation we had before? It turns out that they are related by a similarity transformation and therefore are *equivalent*! Specifically, the third factor of U is related to the third factor of $\tilde{U}$ by $SUS^{-1} = \tilde{U}$:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix} \cdot \begin{pmatrix} \exp -i\theta_z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp i\theta_z \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The same similarity transformation also works for the other factors of U (see *Wikipedia, "3D Rotation Group", footnote 3* for the matrix S). Because the 3-dimensional representation of SU(2) can be made real by choosing the appropriate basis, it is called a *real* representation, even though in some bases its components are complex [GTNut, Ch. II.4]. In fact, all odd-dimensional representations of SU(2) are *real* and all even-dimensional representations are *pseudoreal* [GTNut, Ch. IV.7, p. 275].

We are now in a position to appreciate the power of representation theory: To describe a physical qubit (= the state of a quantum-mechanical two-state system) explicitly, we need to pick a frame (= two basis states); the descriptions arising from all possible frames are related by the transformations of the defining representation of SU(2). To describe a vector in physical 3D space explicitly, we need to pick a coordinate frame; the descriptions arising from all possible Cartesian coordinate frames are related by the transformations of the real 3-dimensional representation of SU(2). If the qubit is realized by a spin-½ particle, then the two descriptions are "connected" in the sense that the qubit amplitudes (for spin up and spin down) and the components of the spin-polarization vector (= direction in 3D space for which the spin is always up) must be transformed jointly using two different representations of the same SU(2) element.

As pointed out earlier, a spin-½ particle needs to be rotated by $720°$ before it returns to its initial state. A simple $360°$ rotation reverses the sign of the state. An object that behaves in this unusual way is called a *spinorial object* and is described by a *spinor*. In contrast, a spin-1 particle or a classical object rotates on a normal $360°$ schedule. Such an object is described by a *vector*. (Note that physicists reserve the term *vector* for arrow-like objects that live in physical 3D space only. In contrast, mathematicians use the term vector for any element that lives in a vector space. Thus, in the mathematical sense, a spinor is also a vector.)

Do spin-½ particles, such as electrons or fermions in general, really behave in this strange way? One way to test this is to interfere a fermion with itself in two different ways: (i) without a rotation between the two "selves" and (ii) with one "self" rotated by $360°$ relative to the other "self". In the first case we should observe constructive interference and in the second case destructive interference. Such experiments have been carried out with neutrons and the expected interference effects were observed (c.f. https://www.researchgate.net/publication/250802309_Fiber_Bundles_and_Quantum_Theory)!

Fermions exhibit another unusual behavior: the exchange of two identical fermions also reverses the sign of the state! (We will come to this later.) According to the *spin-statistics theorem*, the sign change due to an exchange and the sign change due to a $360°$ rotation always appear together: fermions exhibit both sign changes, bosons exhibit neither (cf. *L. Susskind: "Fermions: A tale of two minus signs",* http://theoreticalminimum.com/courses/advanced-quantum-mechanics/2013/fall/lecture-5).

Remarkably, classical objects that are attached to a wall by one or more ribbons behave like spinorial objects. If we rotate such an object by $360°$ the ribbon(s) become twisted, but if we continue to rotate by another $360°$ the twists in the ribbon(s) can be removed! Search the Internet for cool animations. A variation of this trick is the Philippine "wine dance", in which the arm that connects the hand (holding the wine glass) to the body plays the role of the ribbon. While this topological property is interesting, it doesn't really explain why fermions behave that way. They don't seem to have such attachments, or do they?