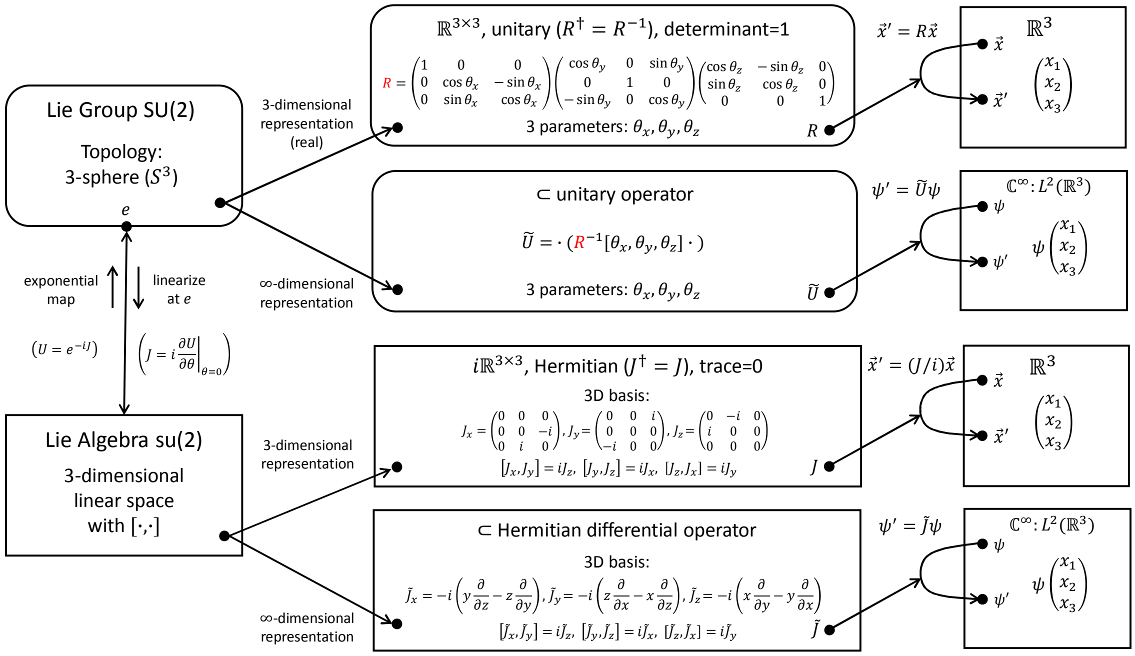
3.11 SU(2): Infinite-Dimensional Representations; Functions and Differential Operators



After exploring several finite-dimensional representations, we now turn to an infinite-dimensional representation of SU(2). Instead of letting the representation act on a column vector, we now let it act on a *function* of a continuous variable, $\psi(x)$. We can think of this function as a vector with infinitely many entries, one for each possible value of the function's argument x. There are many kinds of functions, and we must specify which one we are talking about. Here we choose the square-integrable complex functions of three real variables: $\psi(\vec{x}) \in L^2(\mathbb{R}^3)$ where $\vec{x} = (x_1, x_2, x_3)^T = (x, y, z)^T$. Functions of this kind taper off to zero fast enough such that they can be normalized. This feature makes them suitable for quantum-mechanical wave functions. (Our $\psi(\vec{x})$ could represent a single-particle wave function at one instant of time; in Dirac notation such a wave function would be written $\langle \vec{x} | \psi \rangle$.)

How does SU(2) act on these functions? The upper branch of the diagram shows again the 3-dimensional representation on real vectors from a few examples ago, but with the transformation matrix U renamed to R (for rotation). Now, we use this 3-dimensional representation to transform the 3D argument $\vec{x}$ of our function: $\psi'(\vec{x}) = \psi(R^{-1}\vec{x})$. In other words, we reuse the old "mechanism" for rotating vectors in 3D space, but now we "attach" a function to it. (Somewhat like attaching a flight-simulator cabin to an existing hydraulic mechanism.) We must use the inverse matrix, R^{-1}, rather than R, because to rotate a function one way, we need to rotate its argument in the opposite way [QTGR, Ch. 1.3.2]. (For a simple analogy: to shift the function $f(x)$ by one unit in the positive x direction, we need to write $f(x-1)$.)

While it was straightforward to separate a matrix from the vector it acts on, it is not obvious how to separate the operator in the expression $\psi(R^{-1}[\theta_x, \theta_y, \theta_z]\,\vec{x})$ from the function $\psi(\vec{x})$ it acts on. We could simply define the operator $\tilde{U}$ such that $\tilde{U}\,\psi(\vec{x}) := \psi(R^{-1}[\theta_x, \theta_y, \theta_z]\,\vec{x})$. But here we use the less formal "dot notation" and write $\tilde{U} = \cdot\,(R^{-1}[\theta_x, \theta_y, \theta_z]\,\cdot)$. When this operator acts on the function $\psi(\vec{x})$, the function's name ψ gets inserted at the first dot and the argument $\vec{x}$ gets inserted at the second dot. The lower branch of the diagram shows our infinite-dimensional representation acting on functions.

To find the elements of the Lie algebra, we have to differentiate the elements of the Lie group. This was easy to do when the elements were matrix operators, but our new operator $\widetilde{U}$ is closely intertwined with the function it acts on. The trick is to differentiate them together and then split the result into a differential operator $\tilde{J}$ and the function it acts on. Focusing on the parameter θ_z, the operator acting on the function can be written explicitly as

$$\widetilde{U}(\theta_z)\,\psi(\vec{x}) = \psi(R^{-1}[\theta_z]\,\vec{x}) = \psi(R[-\theta_z]\,\vec{x}) = \psi\begin{pmatrix} x_1\cos\theta_z + x_2\sin\theta_z \\ -x_1\sin\theta_z + x_2\cos\theta_z \\ x_3 \end{pmatrix}.$$

Taking the derivative with respect to θ_z (using the chain rule), setting θ_z to zero, and multiplying by i yields the operator acting on the function

$$\tilde{J}_z\,\psi(\vec{x}) = i\left(\frac{\partial\psi}{\partial x_1}\cdot x_2 + \frac{\partial\psi}{\partial x_2}\cdot(-x_1)\right) = i\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}\right)\psi.$$

Now, splitting off the operator from the function, we get the basis generator

$$\tilde{J}_z = i\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}\right).$$

The generators $\tilde{J}$ of our infinite-dimensional representation are *differential operators*! We can think of a differential operator as a (sparse) matrix with infinitely many entries. The lower branch of the diagram shows the explicit form of all three basis generators, where we have written x, y and z instead of x_1, x_2 and x_3 to get rid of the indices. To make the notation more compact, we can collect the three basis generators into a vector, $\vec{J} := (J_x, J_y, J_z)^T$, and write $\vec{J} = -i\vec{x}\times\vec{\nabla}$, where $\times$ is the cross product and $\vec{\nabla} = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)^T$. Finally, it can be shown that our new basis generators satisfy the usual commutation relations of su(2) [GTNut, Ch. I.3 App. 1].

Are the group elements $\widetilde{U}$ unitary and the generators $\tilde{J}$ Hermitian, just like their finite-dimensional counterparts U and J were? To answer these questions, we first need to understand what it means for an operator that acts on functions (as opposed to on vectors) to have these properties. We can't check $U^\dagger U = I$ for unitarity, as we did for matrix operators! Instead, we must let the operator act on a function, $\widetilde{U}\psi(x)$, and then check its properties using the Hermitian inner product of two functions $\psi(x)$ and $\phi(x)$, which is defined as $\int \psi^*(x)\phi(x)\,dx$ (in Dirac notation this product would be written $\langle\psi|\phi\rangle$). Fortunately, our functions are square integrable!

To check whether an operator is unitary, we test if $\int \left[\widetilde{U}\psi(x)\right]^* \widetilde{U}\phi(x)\,dx = \int \psi^*(x)\phi(x)\,dx$ holds for any pair of functions $\psi(x)$ and $\phi(x)$. (This is analogous to the matrix test $(Ux)^\dagger Uy = x^\dagger y$ from which $U^\dagger U = I$ follows.) Similarly, to check whether an operator is Hermitian, we test if $\int \psi^*(x)\widetilde{H}\phi(x)\,dx = \int [\widetilde{H}\psi(x)]^*\phi(x)\,dx$ holds for any pair of functions $\psi(x)$ and $\phi(x)$. (This is analogous to the matrix test $x^\dagger Hy = (Hx)^\dagger y$ from which $H = H^\dagger$ follows.) It turns out that indeed $\widetilde{U}$ is unitary and $\tilde{J}$ is Hermitian.