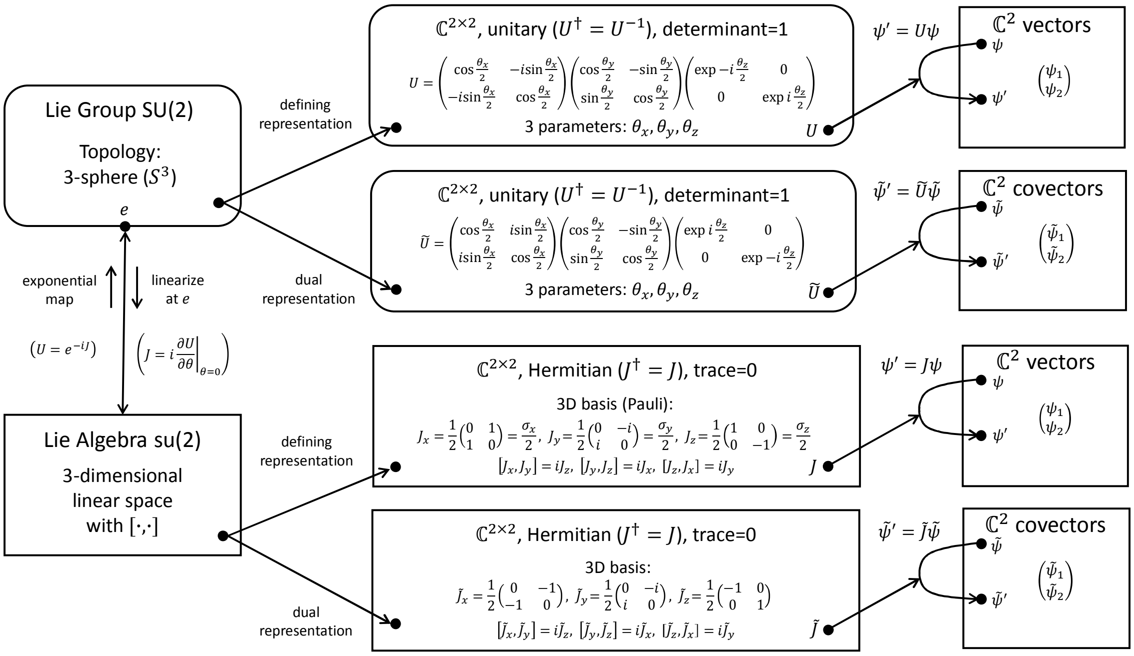
3.7 SU(2): Dual and Complex-Conjugate Representations; Similarity Transformations



Whenever we have a representation of a group, we can construct its *dual representation*. We do that by letting the representation act on the *dual vector space.* In the following, we apply this idea to the defining representation of SU(2), which is shown again in the upper branch of the diagram.

What is the dual vector space? It is the space of scalar linear functions acting on the vectors in the original vector space. Given the complex 2-component vector $\psi = (\psi_1, \psi_2)^T$, the dual vector, which is also known as *covector,* is the scalar linear function $f_{\tilde{\phi}}(\psi) = \tilde{\phi}_1\psi_1 + \tilde{\phi}_2\psi_2$. This function is characterized by two complex numbers, $\tilde{\phi}_1$ and $\tilde{\phi}_2$, which we can pack into another column vector, $\tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2)^T$, where we used a tilde to distinguish covectors from vectors. With this notation, the scalar linear function acting on a vector can be written as $f_{\tilde{\phi}}(\psi) = \tilde{\phi}^T\psi$.

Now, if we know how a representation acts on vectors, we can figure out how it acts on covectors. The defining representation of SU(2) acts on vectors like $\psi' = U\psi$. A covector acting on a vector yields the scalar $\tilde{\phi}^T\psi$, which, by definition, remains invariant under U. Combining these two facts, we conclude that $\tilde{\phi}^T\psi = \tilde{\phi}^T(U^{-1}U)\psi = (\tilde{\phi}^T U^{-1})(U\psi) = \tilde{\phi}'^T\psi'$ and therefore $\tilde{\phi}'^T = \tilde{\phi}^T U^{-1}$ or, after transposing, $\tilde{\phi}' = U^{-1T}\tilde{\phi}$. Thus, the matrix acting on covectors is the transpose of the inverse of the matrix acting on vectors: $\tilde{U} = U^{-1T}$. Note that the group structure is preserved when passing from the original to the dual representation: $(UV)^{-1T} = U^{-1T}V^{-1T}$. The explicit form of the matrix $\tilde{U}$ is shown in the lower branch of the diagram. Note that the matrices are easy to invert because $U^{-1}(\theta_i) = U(-\theta_i)$.

In the above discussion we chose to represent covectors as column vectors, but we could also represent them as *row vectors*. In that case, covectors act on vectors like $\hat{\phi}\psi$ (no transposition needed) and covectors transform like $\hat{\phi}' = \hat{\phi}U^{-1}$, where we marked the row covectors with a hat instead of a tilde. A common notation for vectors and covectors in quantum mechanics is Dirac's bra-ket notation in which vectors are written as "kets", $|\psi\rangle$, and covectors as "bras", $\langle\phi|$. Bras act on kets by forming bra-kets,

$\langle\phi|\psi\rangle$, kets transform like $|\psi'\rangle = U|\psi\rangle$, and bras transform like $\langle\phi'| = \langle\phi|U^{-1}$. Yet another notation is the tensor-index notation in which vectors are written with upstairs indices, ψ^i, and covectors with downstairs indices, ϕ_i. Now, covectors act on vectors like $\phi_i\psi^i$, vectors transform like $U^i_j\psi^j$, and covectors transform like $U_i^{\ j}\phi_j$, where summations over repeated indices are implied (Einstein summation convention) [GTNut, Ch. IV.4].

Up to this point we did not make use of the fact that U is unitary. Covectors transform under $\widetilde{U} = U^{-1T}$ for any (invertible) vector transformation matrix U. But if U is unitary, we have $U^\dagger = U^{-1}$ and thus $\widetilde{U} = U^*$, where the star indicates complex conjugation. Thus, in the unitary case the dual representation is also the *complex-conjugate representation*. Furthermore, the Hermitian inner product between two vectors, $\phi^\dagger\psi$, which remains invariant under a unitary transformation, permits us to *identify* covectors with vectors. Comparing $\tilde{\phi}^T\psi$ with $\phi^\dagger\psi$, we realize that the two scalar results agree for any vector ψ, if we identify the covector $\tilde{\phi}$ with the vector ϕ^*. Thus, in the case of unitary representations, covectors can be understood as the complex conjugate of vectors, $\tilde{\phi} = \phi^*$. (Row-vector notation: $\hat{\phi} = \phi^\dagger$; Dirac notation: $\langle\phi| = |\phi\rangle^\dagger$; tensor-index notation: $\phi_i = \phi^{i*}$.)

Is the dual (or complex-conjugate) representation really a *new* representations of SU(2)? Two representations are considered *equivalent* if they are related by a *similarity transformation*. Specifically, for any representation that acts like $\psi' = U\psi$, we can construct an equivalent (isomorphic) representation $\overline{U}$ that acts on the same vector in a different basis like $\phi' = \overline{U}\phi$, where $\phi = S\psi$ and S is a fixed invertible matrix. Expanding $\phi' = \overline{U}\phi$ to $S\psi' = \overline{U}S\psi$ and left multiplying with S^{-1} yields $\psi' = S^{-1}\overline{U}S\psi$ and thus $U = S^{-1}\overline{U}S$ or $\overline{U} = SUS^{-1}$, which is known as a similarity transformation. As expected, the group structure is preserved under such a transformation: $S(UV)S^{-1} = SUS^{-1}SVS^{-1}$. (A fixed parametrization of the group elements is assumed here. The freedom to choose different parameters, discussed earlier, and the freedom to choose different bases are two different things.)

Thus, the question is: can we map a general SU(2) matrix to its complex conjugate with a similarity transformation: $SUS^{-1} = U^*$? Yes, we can, although finding the appropriate S may be a bit tricky:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_0 - ia_z & -a_y - ia_x \\ a_y - ia_x & a_0 + ia_z \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_0 + ia_z & -a_y + ia_x \\ a_y + ia_x & a_0 - ia_z \end{pmatrix},$$

where we wrote the SU(2) matrix in the "3-sphere parameter" form introduced earlier and S equals the 2-dimensional Levi-Civita symbol ε_{ij} [GTNut, p. 252]. The same similarity transformation also maps the generators of the defining representation to their complex conjugate, if we divide out the factor i first: $S(J/i)S^{-1} = (J/i)^*$. Thus, we conclude that the defining and the complex-conjugate representations are *equivalent*. In fact, SU(2) has only one irreducible 2-dimensional representation!

As we will see later, taking the complex conjugate of a complex representations generally *does* result in a new, inequivalent representation. A representation that is equivalent to its complex conjugate, like the defining representation of SU(2) discussed above, is called *pseudoreal* [GTNut, Ch. IV.5]. Why does this happen for an SU(2) matrix? Similarity transformations do *not* change the eigenvalues of the matrix they act on. So, given a general set of complex eigenvalues, it is not possible to map them all to their complex conjugate [QTGR, Ch. 41.1]. But for an SU(2) matrix the two eigenvalues are special: they are complex conjugates of each other! In this case it *is* possible to find a similarity transformation that maps the matrix to its complex conjugate.