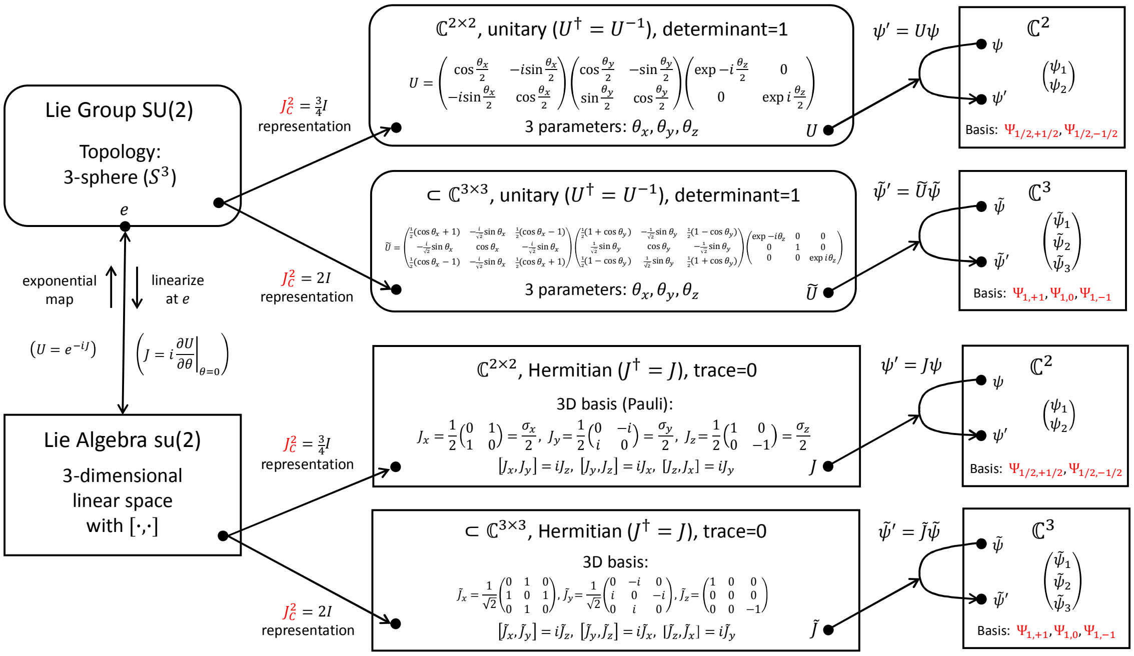
3.10 SU(2): Labels for Basis Vectors and Representations; Casimir Operator



Now that we have seen a couple of representations of SU(2), let's think more systematically about (i) finding a meaningful basis for the representation space, (ii) labeling the basis vectors of this space, and (iii) labeling the representations themselves. This may seem like a rather pedantic thing to do for the representations we have encountered so far, but it will become important in later examples.

All the representations of SU(2) that we have seen are unitary and the corresponding representations of the algebra, su(2), are Hermitian (after multiplication by i). This is a general feature of topologically *compact* Lie groups, that is, groups that don't "go off to infinity" [GTNut, Ch. II.1]. Hermitian matrices have a complete set of orthogonal eigenvectors, that is, a set that spans the entire vector space they act on [TM, Vol. 2, Ch. 3.1.5]. It is a good idea to choose these eigenvectors as the *basis vectors* for the representation space. Conveniently, each eigenvector has a (real) eigenvalue associated with it that can be used to *label* the basis vector. (In general, the eigenvalues of several commuting matrices are needed to *uniquely* label a basis vector. However, the su(2) algebra has no commuting matrices and so one eigenvalue is enough.) Why is this meaningful? In quantum mechanics, eigenvectors (= our basis vectors) represent quantum states for which an observable has a *definite value* and eigenvalues (= our labels of basis vectors) represent the possible values the observable can have.

Let's try this out for the defining representation of su(2). The basis generator $J_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has an eigenvector $\Psi_{+1/2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with eigenvalue +½ and another eigenvector $\Psi_{-1/2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigenvalue −½. Note that we used the eigenvalue as a subscript to label the eigenvector. The eigenvectors for the basis generator $J_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ are $\Psi_{+1/2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\Psi_{-1/2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ and, finally, for $J_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ they are $\Psi_{+1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\Psi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In all cases the eigenvalues are +½ and −½. We could use any of these eigenvector pairs as the basis for our representation space, but it is conventional to choose those of J_z. These basis vectors correspond to the states with definite spin up and definite spin down (along the z axis) and we'll write them as $\Psi_{+1/2}$ and $\Psi_{-1/2}$ (in Dirac notation these states would be written as $|+½\rangle$ and $|-½\rangle$). The same procedure can be applied to the 3-dimensional representation of su(2). In that case, the basis vectors are $\Psi_{+1} = (1, 0, 0)^T$, $\Psi_0 = (0, 1, 0)^T$, and $\Psi_{-1} = (0, 0, 1)^T$ (in Dirac notation these states would be written as $|+1\rangle$, $|0\rangle$, and $|-1\rangle$). Choosing and labeling a basis in this formal way seems unnecessarily complicated: we would have picked this simple basis anyway! However, this method will become important when we come to infinite-dimensional and tensor-product representations.

Next, we introduce the *Casimir operator*. The defining property of the Casimir operator is that it commutes with all the elements of a Lie algebra [PfS, Ch. 3.5]. To do so, it must commute with all the basis generators. For su(2), the quadratic expression $J_C^2 = J_x^2 + J_y^2 + J_z^2$ is a Casimir operator, that is, $[J_C^2, J_i] = 0$ for all i. Let's try this out for the defining representation. Squaring the basis generators, we find $J_C^2 = J_x^2 + J_y^2 + J_z^2 = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3}{4}I$, which is just ¾ times the identity matrix. No wonder it commutes with all the elements of the Lie algebra! For the 3-dimensional representation, we get $J_C^2 = J_x^2 + J_y^2 + J_z^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2I$, that is, two times the identity matrix.

Now, representation theory (Schur's lemma) says that we can use the eigenvalues, c, of the Casimir operator to delineate and label the vectors belonging to different irreducible representations [PfS, Ch. 3.5]. Applied to the above example, we find that all the vectors of the 2-dimensional representation are eigenvectors of the Casimir operator with eigenvalue $c = 3/4$ and that all the vectors of the 3-dimensional representation are eigenvectors of the Casimir operator with eigenvalue $c = 2$. As advertised, the Casimir operator delineates the vectors of the two irreducible representations by assigning different labels (eigenvalues) to them. Of course, we could just as well have used the vector dimension to delineate and label the two representations (as we have done up to now). The power of the Casimir operator will become apparent when we get to *reducible* representation, which consist of several irreducible representations "mixed together".

Is there a relationship between the eigenvalue c of the Casimir operator J_C^2, and the dimension d of the representation? It turns out that $c = j(j + 1)$, where j is the spin (label) of the representation, which is $j = (d - 1)/2$, as we know [PfS, Ch. 3.6.1]. So, for the 2-dimensional representation, we have $j = 1/2$ and thus $c = 1/2\ (1/2 + 1) = 3/4$ and for the 3-dimensional representation, we have $j = 1$ and thus $c = 1\ (1 + 1) = 2$. It checks out!

In conclusion, we introduced two labels for each basis vector, one labeling the *irreducible representation* the basis vector belongs to and one labeling the *basis vector* itself within that representation. The first label could be the eigenvalue of the Casimir operator, $j(j + 1)$, but it is conventional to just use j, which represents the amount of total spin. The second label is the eigenvalue of J_z, conventionally called m (earlier we called this j_i), which represents the spin component in the z direction. Therefore, a general basis vector is written with two indices: $\Psi_{j,m}$ (in Dirac notation such a basis vector would be written as $|j, m\rangle$). For example, see the (red) basis vectors shown in the diagram. Now we have the necessary tools in hand to tackle more complicated representations!