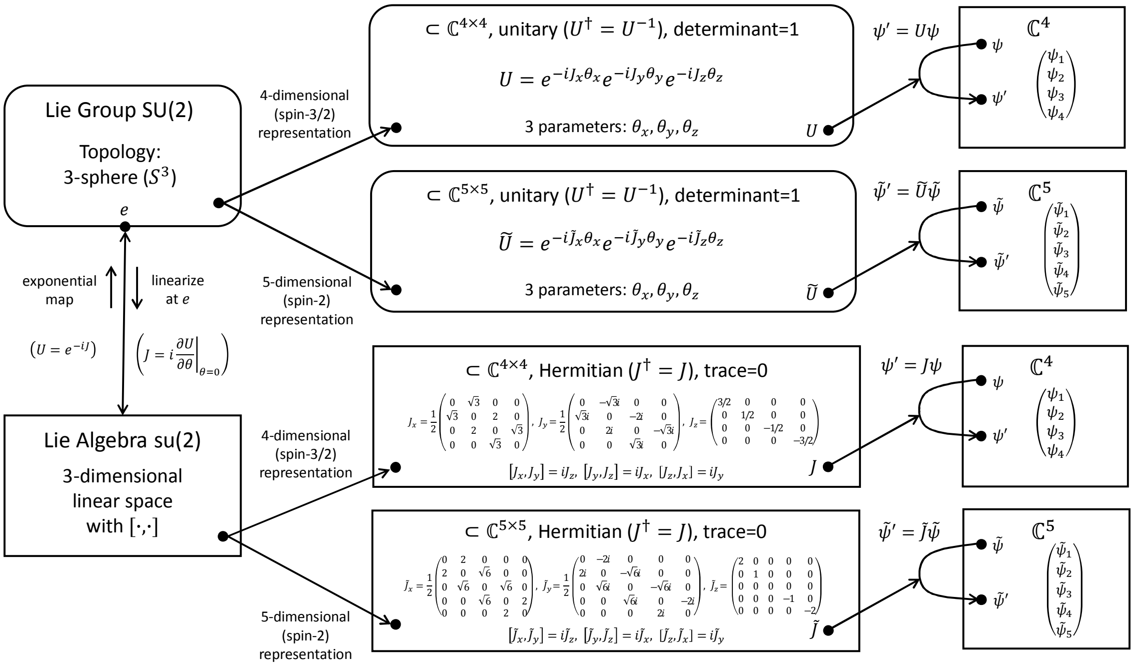
3.3 SU(2): Four- and Five-Dimensional Representations; Higher Spins



As mentioned earlier, SU(2) has representations in all dimensions from one to infinity. We already met the 2- and 3-dimensional representations. Now, we have a look at the 4- and 5-dimensional representations (see the diagram). All these representations are so-called *irreducible* representations, that is, representations that cannot be broken up into smaller ones. These representations are of particular importance because they provide the building blocks of representation theory. Later we will encounter *reducible* representations, which are direct sums of irreducible representations.

Applied to quantum-mechanical spin, the 4- and 5-dimensional irreducible representations describe massive spin-3/2 and spin-2 particles, respectively. For example, the delta baryons, which consist of three spin-½ quarks, are spin-3/2 particles. At this time, there are no known elementary particles with spin 3/2. Similarly, massive spin-2 particles occur only as composite objects. (The graviton, which is an elementary spin-2 particle, is *massless* and therefore not described by the above 5-dimensional representation. Later we will have more to say about the spin states of massless particles.)

Let's examine the three basis generators of the 4- and 5-dimensional representations shown in the diagram. It is convenient to choose the basis of the representation space such that as many basis generators as possible become diagonal, thus explicitly showing their eigenvalues on the diagonal. However, for SU(2) this can be done for only *one* basis generator because no pair of basis generators commutes. It is customary to choose J_z for this purpose. Furthermore, note that the basis generator J_x has only real entries, arranged just above and below the main diagonal. Finally, the basis generator J_y has the same entries as J_x except that the entries above the main diagonal are multiplied by $-i$ and those below the main diagonal by i. (This pattern makes $J_x \pm iJ_y$ act like raising and lowering operators. For more details, see the Appendix "The Ladder Trick; Raising and Lowering Operators".)

The explicit transformation matrices $U(\theta_x, \theta_y, \theta_z)$ for these higher-dimensional representations are rather complicated and therefore not shown in the diagram. Instead, we write the transformation matrices as a product of three exponentials. The last factor of the overall transformation, $U(\theta_z) = e^{-iJ_z\theta_z}$, however, is easy to figure out because the generator in the exponent is diagonal. For dimensions two, three, four, and five, we obtain the following matrices:

$$U(\theta_z) = \begin{pmatrix} \exp -i\theta_z/2 & 0 \\ 0 & \exp i\theta_z/2 \end{pmatrix}, \qquad \begin{pmatrix} \exp -i\theta_z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp i\theta_z \end{pmatrix},$$

$$\begin{pmatrix} \exp -i3\theta_z/2 & 0 & 0 & 0 \\ 0 & \exp -i\theta_z/2 & 0 & 0 \\ 0 & 0 & \exp i\theta_z/2 & 0 \\ 0 & 0 & 0 & \exp i3\theta_z/2 \end{pmatrix}, \quad \begin{pmatrix} \exp -i2\theta_z & 0 & 0 & 0 & 0 \\ 0 & \exp -i\theta_z & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \exp i\theta_z & 0 \\ 0 & 0 & 0 & 0 & \exp i2\theta_z \end{pmatrix}.$$

How are we to think about quantum-mechanical spin? Unlike classical spin, it does *not* have to do with speed of rotation! Quantum spin exists independent of time. Instead, it has to do with the amount of rotation that is needed for the quantum state to return to its starting value. Let's assume that our particle is in a state of definite and maximum spin, $\psi = (1, 0, \cdots, 0)^T$, with respect to the z axis (i.e., the particle's spin axis is aligned with the z axis). Now, from the above 3×3 matrix for $U(\theta_z)$ we can see that if we rotate a spin-1 particle by 360° ($\theta_z = 2\pi$), its state is back to where it started. Good. Next, we rotate a spin-2 particle and find that its state is back to where it started after 180° (see the above 5×5 matrix). Somehow, the spin-2 particle has a bilateral symmetry like a playing card: we can't distinguish if it's rotated by 180° or not at all. What about a spin-½ particle? Rotating it by 360° doesn't bring its state back to where it started, we need to rotate it by 720° (see the above 2×2 matrix)! This is decidedly weird, and we will come back to this phenomenon later. In summary, given a particle with spin j, we need to rotate it by $360°/j$ to bring its state back to where it started. Formulated differently, a particle with spin j goes through j wave periods as we rotate it around by 360°. In some sense, quantum-mechanical spin is a *measure of angular waviness*.

Can we change the spin of an elementary particle? Could we blast an electron with enough energy to change its spin from ½ to 1? No, we can't. The problem is that the energy required to get to the next level of spin is so large that the electron would no longer be an electron! Classically, the energy needed to increase the spin of an object by $\Delta S = \hbar/2$ is $\Delta E = (\Delta S)^2/(2I) = \hbar^2/(8I)$, where I is the moment of inertia. The latter is of the form $I \propto mr^2$, where r is related to the radius of the spinning object. So, for a point-like elementary particle, such as an electron, I is pretty much zero and thus the energy, ΔE, needed to spin it up goes to infinity! In contrast, composite particles, such as mesons and hadrons, which are made up of quarks connected by rubber-band-like gluon strings, *can* be spun up. (This paragraph is based on *L. Susskind: "Angular momentum",* https://theoreticalminimum.com/courses/new-revolutions-particle-physics-basic-concepts/2009/fall/lecture-7).

In conclusion, to describe a particle with total spin j, we use the k-dimensional irreducible representation of SU(2), where $k = 2j + 1$. The generator J_z of that representation is a $k \times k$ matrix with k distinct eigenvalues. Thus, the z-component of the spin of such a particle assumes one of k possible values upon measurement: the higher the total spin j, the larger the set of possible measurement outcomes for its spin components. A particle with total spin j needs to be rotated by $360°/j$ about its spin axis before its state returns to where it started.