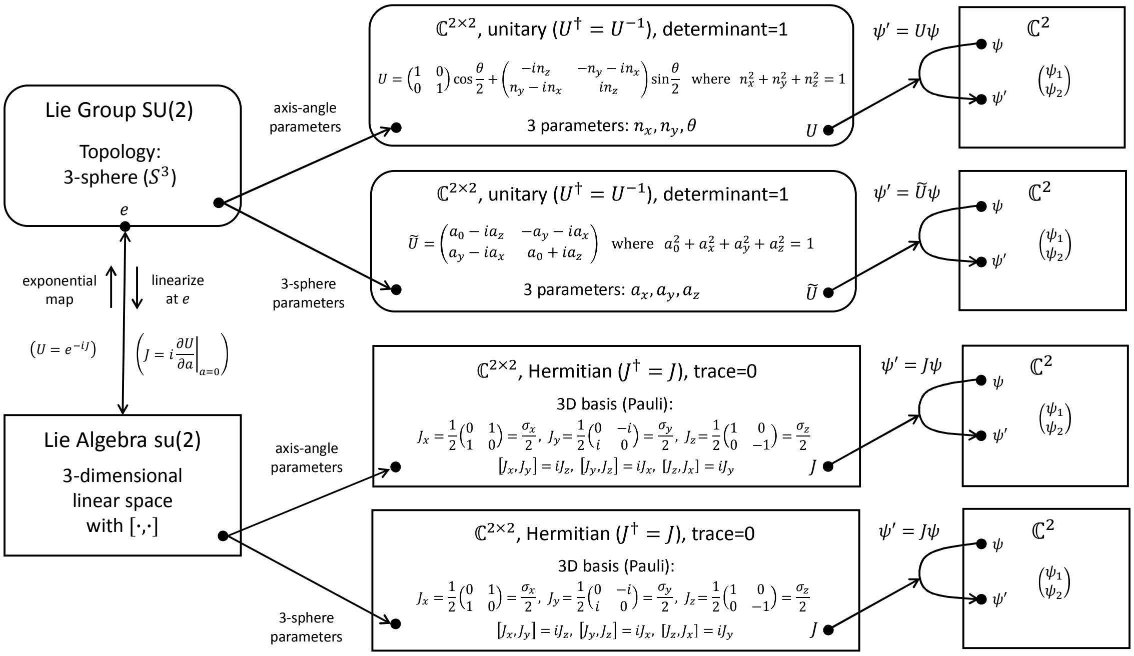
3.5 SU(2): Defining Representation with 3-Sphere Parameters; Group Topology



Let's explore one more parametrization of the same SU(2) matrix. The upper branch of the diagram shows again the matrix expression for the axis-angle parametrization from the previous example, whereas the lower branch shows an expression that is based on the new parameters $a_0 = \cos\frac{\theta}{2}$, $a_x = n_x \sin\frac{\theta}{2}$, $a_y = n_y \sin\frac{\theta}{2}$, $a_z = n_z \sin\frac{\theta}{2}$. Using the fact that $\sin^2\phi + \cos^2\phi = 1$, the constraint $n_x^2 + n_y^2 + n_z^2 = 1$ of the axis-angle parametrization translates to $a_0^2 + a_x^2 + a_y^2 + a_z^2 = 1$ for the new parametrization. The transformation shown in the diagram can be rewritten more compactly in terms of Pauli matrices as

$$\tilde{U}(a_0, a_x, a_y, a_z) = a_0 I - i(a_x\sigma_x + a_y\sigma_y + a_z\sigma_z)$$

or, using vector notation, as $\tilde{U}(a_0, \vec{a}) = a_0 I - i(\vec{a}\cdot\vec{\sigma})$, where $\vec{a} = (a_x, a_y, a_z)^T$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$.

Just like in the previous example, both parametrizations describe the same *set* of transformation matrices and each element of the set lives in the *same* neighborhood. It's the *same* representation parametrized in two different ways!

This representation has again three free parameters. Somewhat sloppily, we listed them in the diagram as a_x, a_y, and a_z implying that a_0 is determined by the constraint (this is not exactly true because the sign of a_0 remains ambiguous). The identity transformation is now given by $a_0 = 1$, $a_x = 0$, $a_y = 0$, and $a_z = 0$ (without ambiguity). Calculating the derivatives (times i) with respect to a_x, a_y, a_z and evaluating them at the mentioned parameter values results in the basis generators σ_x, σ_y, and σ_z (no tricks required). After normalizing them such that their eigenvalues become +½ and −½, we arrive at our usual basis generators: $J_x = \sigma_x/2$, $J_y = \sigma_y/2$, $J_z = \sigma_z/2$.

What do we learn from this new parametrization? The a_k parameters do not represent rotation angles. But the fact that $a_0^2 + a_x^2 + a_y^2 + a_z^2 = 1$ means that the valid points in the parameter space form a *3-*

sphere, S^3, which is the 4-dimensional generalization of a regular sphere (= 2-sphere), S^2. Because the group elements depend smoothly on these parameters, the group manifold also has the *topology of a 3-sphere*. This is true regardless of the chosen parametrization: it is a property of the (abstract) group SU(2). We can think of the different parametrizations as different ways to put coordinates on the group manifold.

How can we visualize the topology of a 3-sphere? Let's take a step back and first consider the 2-sphere, that is, the regular sphere. An inflated rubber balloon is a good example of a 2-sphere. Now, let's deflate the balloon, which will change its shape but not its topology. We end up with two round rubber sheets (= rubber disks) on top of each other, which are connected at their edges. Similarly, we can picture the topology of a 3-sphere as two 3D balls that are connected (identified) at their surfaces. This means that when we come out of one ball at a particular point on the surface, we immediately enter the other ball at the corresponding point. To make the identification of the two surfaces more natural, we may imagine the two balls as being "inside each other", occupying the same space, but only being connected at the surface. That's not easy to visualize, but we may picture a worm-eaten apple with two dense networks of holes that connect only at the surface of the apple (see Fig. 18 in *George Gamow: "One Two Three … Infinity: Facts and Speculations of Science"* for an illustration).

Another way to visualize a 3-sphere is to imagine it passing through a 3D space such that we get a time sequence of 3D cuts of the 3-sphere. Let's first do this with a 2-sphere passing through a 2D space: At first, we see just a dot. Then, the dot turns into a tiny circle that keeps on growing until it reaches its maximum size. Then, the circle starts shrinking again and eventually becomes just a dot and disappears. Now, returning to the 3-sphere passing through the 3D space: At first, we see just a dot. Then, the dot turns into a tiny 2-sphere that keeps on growing until it reaches its maximum size. Then, the 2-sphere starts shrinking again and eventually becomes just a dot and disappears. If we imagine this sequence of 2-spheres layered on top of each other, as if printed out with a 3D printer, we end up with the two balls discussed above!

Yet another way to visualize a 3-sphere is to cut it (take the intersection) with a (2-dimensional) plane that passes through the center of the 3-sphere. Let's first try this out on a 2-sphere. If we cut a regular sphere in this way, we get (great) circles, which are all of the same size. Moreover, any two of these circles are connected at two points. Now, if we cut a 3-sphere in this way, we also get circles of the same size, but they are all completely *disconnected*. Any two of these circles twist around each other to form a link. The cutting up of a 3-sphere into such circles is known as a *Hopf fibration* [RtR, Ch. 15.4]. Mathematicians say that S^3 is an S^1 bundle over S^2, where S^1 is a circle (= 1-sphere), which represents one of the fibers, and S^2 is the 2-sphere, which corresponds to all possible ways the plane can be placed in the 4D space. Helpful illustrations and videos can be found on the Internet.

The 3-sphere, just like the regular sphere, "closes back in on itself" rather than going "off to infinity". It is said that the 3-sphere is a *compact* space. In contrast, Euclidean space is noncompact. Moreover, any loop embedded in the 3-sphere can be smoothly contracted to a point without getting "stuck". It is said that the 3-sphere is a *simply connected* space (the same is true for the regular sphere). In contrast, the torus is not simply connected because a loop that wraps around it is permanently stuck. Both of these topological features play an important role in the representation theory of SU(2).