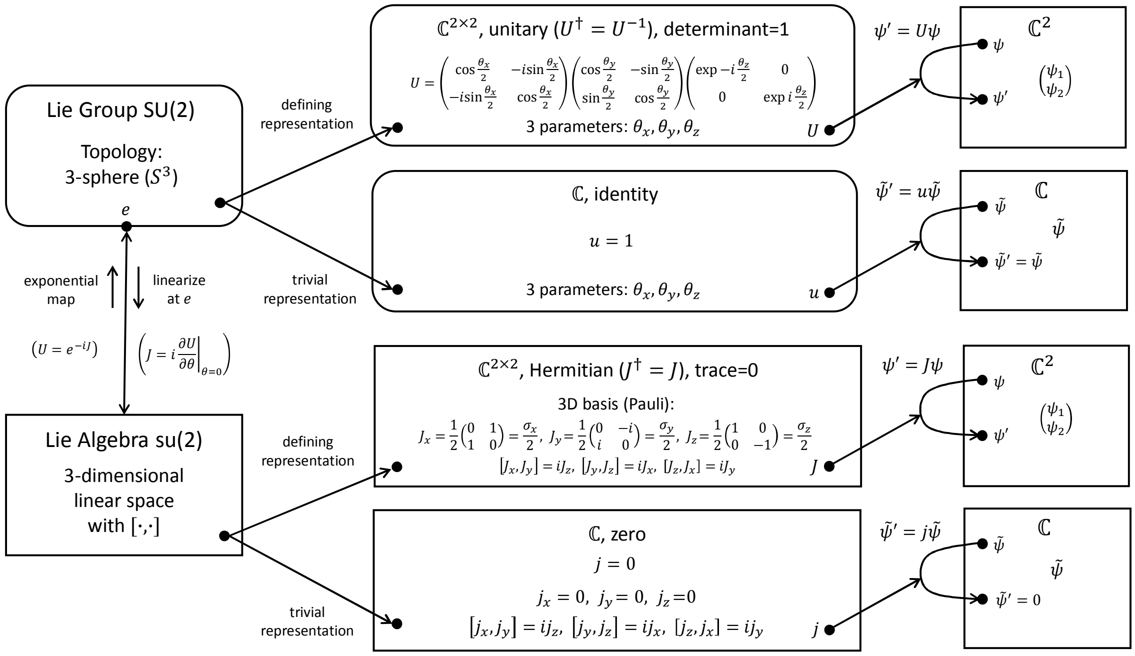
3.6 SU(2): Trivial Representation; Invariants



We looked at the 2-, 3-, 4-, and 5-dimensional representations of SU(2), but we skipped over the 1-dimensional one. What does it do? It acts on 1-dimensional vectors, commonly known as *scalars*. The group and algebra elements are represented by 1×1 matrices, which, again, are scalars. The upper branch of the diagram shows again the defining representation, which acts on 2-dimensional complex vectors, and the lower branch shows the 1-dimensional representation, a.k.a. the *trivial representation*, which acts on (complex) scalars.

What are the algebra elements of this representation? As usual, the basis generators need to satisfy the commutation relations of su(2). But because scalars commute, the only way this can happen is if all three basis generators are zero: $j_x = j_y = j_z = 0$! Thus, we find that the only generator in this representation is $j = 0$.

Exponentiating this generator gives us the group element: $u = e^0 = 1$. Thus, there is only one transformation in this group representation, namely the identity transformation! There is no dependence on any of the three parameters θ_x, θ_y, or θ_z. Such a representation, which maps all group elements to the identity transformation, is called *trivial* and exists for any group. Note that a many-to-one map of this kind is allowed as long as it preserves the group structure, which it does.

All this sounds rather silly, but the trivial representation is a valid and meaningful representation. If we interpret the generator j as the operator for the spin observable of a spin-0 particle, we find that the spin (components) can assume only one value, namely zero (the only eigenvalue of $j = 0$). In other words, the measured spin is always zero, regardless of the orientation of the z axis along which it is measured. The trivial representation of SU(2) correctly describes the spin of *spin-0 particles*, a.k.a. *scalar particles*. The Higgs boson is an example of this kind of particle.

All *invariants* transform under the trivial representation. For example, we know that given two spin states ψ and ϕ of a spin-½ particle, the Hermitian inner product $\phi^\dagger\psi = \phi_1^*\psi_1 + \phi_2^*\psi_2$, a.k.a. the *overlap*, is independent of the chosen basis states (coordinate frame). This makes sense because the overlap can be interpreted as the probability amplitude for state ψ to be in state ϕ. We can thus say that the overlap transforms under the trivial representation of SU(2). Mathematically, for the overlap to be invariant under the transformation U, we need $(U\phi)^\dagger(U\psi) = \phi^\dagger\psi$. Rewriting this condition as $\phi^\dagger U^\dagger U\psi = \phi^\dagger\psi$, we see that it is equivalent to $U^\dagger U = I$. But this is exactly the condition for unitarity of U, hence the overlap is invariant under S**U**(2).

Interestingly, the overlap is not the only invariant that can be constructed from the two spin states ψ and ϕ: the quantity $\phi_1\psi_2 - \phi_2\psi_1$ is also invariant! We can think of this as the "distance" between the two states or the "area" enclosed between the two states: if the states are identical, $\phi = \psi$, the distance as well as the enclosed area are zero (whereas the overlap is maximum). Rewritten in vector-matrix notation, this invariant reads $\phi^T\varepsilon\psi$, where $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the 2-dimensional Levi-Civita symbol, which in this context is also known as the *spinor metric*. To verify this invariance, we need to check if $(U\phi)^T\varepsilon(U\psi) = \phi^T\varepsilon\psi$. Rewriting this condition as $\phi^T U^T\varepsilon U\psi = \phi^T\varepsilon\psi$, we see that it is equivalent to $U^T\varepsilon U = \varepsilon$. But this is exactly the condition for $\det(U) = 1$, hence $\phi^T\varepsilon\psi$ is invariant under **S**U(2). (The determinant of any 2×2 matrix U_{ij} satisfies $\sum_{ij}\varepsilon_{ij}U_{im}U_{jn} = \varepsilon_{mn}\det(U)$ [GTNut, Ch. IV.4]. Rewritten in the form of a matrix product, we have $U^T\varepsilon U = \varepsilon\det(U)$, which simplifies to $U^T\varepsilon U = \varepsilon$ for $\det(U) = 1$.)

The most important invariant in all of physics is the *action*. The action is a real-valued scalar that governs the dynamics of the system under investigation. For classical systems this is realized by the principle of *"least" action* [TM, Vol. 1] and for quantum-mechanical systems by the principle of *sum over histories* as expressed by the Feynman path integral [TM, Vol. 2]. The action must not depend on our choice of coordinates; for example, the origin, orientation, and inertial velocity of our space-time frame cannot matter. For the action to be invariant, it is sufficient (but not necessary) that the *Lagrangian* is invariant. Often, the Lagrangian is a sum of terms that are invariant by themselves. Later, we will encounter concrete examples of Lagrangians that transform under the trivial representation of the relevant symmetry groups.