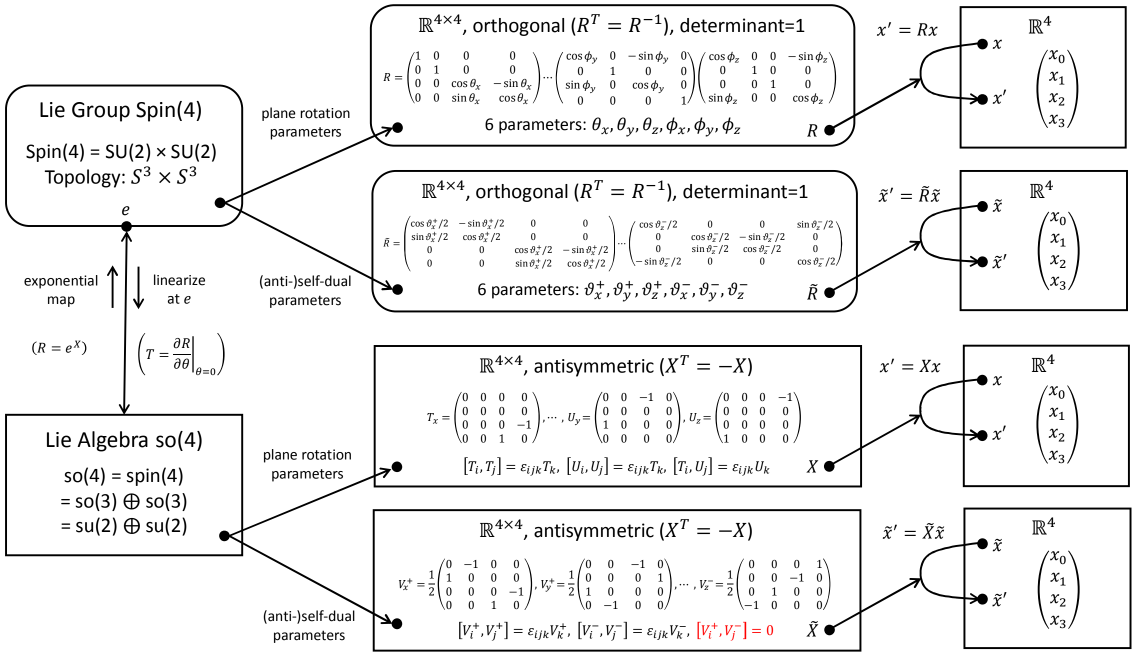
5.5 Spin(4): Representation with (Anti-)Self-Dual Parameters; so(4) = so(3) ⊕ so(3)



Inspired by our experience with the self-dual and anti-self-dual tensor representations (see previous example), we are going to switch to a new set of parameters: instead of the plane-rotation parameters, $\theta_x, \theta_y, \theta_z, \phi_x, \phi_y, \phi_z$, we use the *self-dual double-rotation* parameters, $\vartheta_x^+ = \theta_x + \phi_x$, $\vartheta_y^+ = \theta_y + \phi_y$, $\vartheta_z^+ = \theta_z + \phi_z$, and the *anti-self-dual double-rotation* parameters, $\vartheta_x^- = \theta_x - \phi_x$, $\vartheta_y^- = \theta_y - \phi_y$, $\vartheta_z^- = \theta_z - \phi_z$. The parameter ϑ_x^+, for example, rotates simultaneously in the yz *and* the wx plane by the same amount (angle), whereas the parameter ϑ_x^- rotates by equal and *opposite* amounts in the same two planes. Note that these double rotations no longer rotate about a 2D plane but about a point, namely the origin.

To change the parameters, we start with the original 4D-rotation matrices (see the upper branch of the diagram) and substitute $\theta_i = \frac{1}{2}(\vartheta_i^+ + \vartheta_i^-)$ and $\phi_i = \frac{1}{2}(\vartheta_i^+ - \vartheta_i^-)$ into them. Then, we split each factor into two matrices, one depending only on ϑ_x^+ and the other only on ϑ_x^-. Finally, we combine matrices that depend on the same parameter into a single one. A general 4D rotation can now be written as the product $\tilde{R} = R_x^+(\vartheta_x^+) \cdot R_y^+(\vartheta_y^+) \cdot R_z^+(\vartheta_z^+) \cdot R_x^-(\vartheta_x^-) \cdot R_y^-(\vartheta_y^-) \cdot R_z^-(\vartheta_z^-)$, where

$$R_x^+ = \begin{pmatrix} \cos\vartheta_x^+/2 & -\sin\vartheta_x^+/2 & 0 & 0 \\ \sin\vartheta_x^+/2 & \cos\vartheta_x^+/2 & 0 & 0 \\ 0 & 0 & \cos\vartheta_x^+/2 & -\sin\vartheta_x^+/2 \\ 0 & 0 & \sin\vartheta_x^+/2 & \cos\vartheta_x^+/2 \end{pmatrix}, \; R_x^- = \begin{pmatrix} \cos\vartheta_x^-/2 & \sin\vartheta_x^-/2 & 0 & 0 \\ -\sin\vartheta_x^-/2 & \cos\vartheta_x^-/2 & 0 & 0 \\ 0 & 0 & \cos\vartheta_x^-/2 & -\sin\vartheta_x^-/2 \\ 0 & 0 & \sin\vartheta_x^-/2 & \cos\vartheta_x^-/2 \end{pmatrix},$$

$$R_y^+ = \begin{pmatrix} \cos\vartheta_y^+/2 & 0 & -\sin\vartheta_y^+/2 & 0 \\ 0 & \cos\vartheta_y^+/2 & 0 & \sin\vartheta_y^+/2 \\ \sin\vartheta_y^+/2 & 0 & \cos\vartheta_y^+/2 & 0 \\ 0 & -\sin\vartheta_y^+/2 & 0 & \cos\vartheta_y^+/2 \end{pmatrix}, \; R_y^- = \begin{pmatrix} \cos\vartheta_y^-/2 & 0 & \sin\vartheta_y^-/2 & 0 \\ 0 & \cos\vartheta_y^-/2 & 0 & \sin\vartheta_y^-/2 \\ -\sin\vartheta_y^-/2 & 0 & \cos\vartheta_y^-/2 & 0 \\ 0 & -\sin\vartheta_y^-/2 & 0 & \cos\vartheta_y^-/2 \end{pmatrix},$$

$$R_z^+ = \begin{pmatrix} \cos\vartheta_z^+/2 & 0 & 0 & -\sin\vartheta_z^+/2 \\ 0 & \cos\vartheta_z^+/2 & -\sin\vartheta_z^+/2 & 0 \\ 0 & \sin\vartheta_z^+/2 & \cos\vartheta_z^+/2 & 0 \\ \sin\vartheta_z^+/2 & 0 & 0 & \cos\vartheta_z^+/2 \end{pmatrix}, \; R_z^- = \begin{pmatrix} \cos\vartheta_z^-/2 & 0 & 0 & \sin\vartheta_z^-/2 \\ 0 & \cos\vartheta_z^-/2 & -\sin\vartheta_z^-/2 & 0 \\ 0 & \sin\vartheta_z^-/2 & \cos\vartheta_z^-/2 & 0 \\ -\sin\vartheta_z^-/2 & 0 & 0 & \cos\vartheta_z^-/2 \end{pmatrix},$$

as shown in the lower branch of the diagram. This product can be split into a *self-dual* double rotation $R^+ = R_x^+(\vartheta_x^+) \cdot R_y^+(\vartheta_y^+) \cdot R_z^+(\vartheta_z^+)$ and an *anti-self-dual* double rotation $R^- = R_x^-(\vartheta_x^-) \cdot R_y^-(\vartheta_y^-) \cdot R_z^-(\vartheta_z^-)$. Amazingly, it turns out that these two types of rotations *commute*: $\tilde{R} = R^+ \cdot R^- = R^- \cdot R^+$!

Taking the derivatives of the rotation matrix $\tilde{R}$ with respect to the new parameters and setting them to zero, we find the following basis generators:

$$V_x^+ = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; V_y^+ = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; V_z^+ = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$V_x^- = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; V_y^- = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; V_z^- = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Note that these basis generators are just linear combinations of the old ones: $V_x^+ = \frac{1}{2}(T_x + U_x)$, $V_y^+ = \frac{1}{2}(T_y + U_y)$, $V_z^+ = \frac{1}{2}(T_z + U_z)$ and $V_x^- = \frac{1}{2}(T_x - U_x)$, $V_y^- = \frac{1}{2}(T_y - U_y)$, $V_z^- = \frac{1}{2}(T_z - U_z)$. Both bases span the same space, namely the space of all antisymmetric 4×4 matrices. The diagram shows the original basis generators in the upper branch and the new basis generators in the lower branch.

What are the commutation relations among the new basis generators V_i^+ and V_i^-? Amazingly, all three V_i^+ commute with all three V_i^-, that is, $[V_i^+, V_j^-] = 0$! In other words, the so(4) algebra falls apart into two independent parts [GTNut, Ch. I.3, Appendix 2, p. 83]. Moreover, the commutation relations among the V_i^+ are the same as those among the V_i^-, namely $[V_i^+, V_j^+] = \varepsilon_{ijk} V_k^+$ and $[V_i^-, V_j^-] = \varepsilon_{ijk} V_k^-$. Whereas there were some similarities between the T_i and U_i, the V_i^+ and V_i^- behave identically! Finally, the commutation relations $[V_i^\pm, V_j^\pm] = \varepsilon_{ijk} V_k^\pm$ are exactly those known from so(3). All of this can be summarized by writing so(4) = so(3) ⊕ so(3), that is, the so(4) algebra is the direct sum of two so(3) algebras. (Note that we are now talking about the direct sum of two algebras, not two representations.)

It turns out that the new matrix $\tilde{R}$ is *not* a representation of SO(4) but of Spin(4), its *double cover*. Note that the new parameters $\vartheta_i^\pm$ range from 0 to 720° and that $\vartheta_i^\pm = 0$ and $\vartheta_i^\pm = 360°$ both parametrize the identity. By exponentiating spin(4) = so(4) = so(3) ⊕ so(3) = su(2) ⊕ su(2), we find that Spin(4) is the direct product SU(2) × SU(2), which explains why $\tilde{R}$ can be written as a product of two commuting factors. (The *direct* product and the *direct* sum are both based on the Cartesian product of the underlying sets. The *product* or *sum* in the name refers to the operation defined on these sets.) Finally, note that all representations of SO(4) are also representations of Spin(4).

For additional information about plane rotations and double rotations in four dimensions, see https://twitter.com/johncarlosbaez/status/1290324611222011908, https://en.wikipedia.org/wiki/Rotations_in_4-dimensional_Euclidean_space; for visualizations, see http://eusebeia.dyndns.org/4d/vis/10-rot-1.)