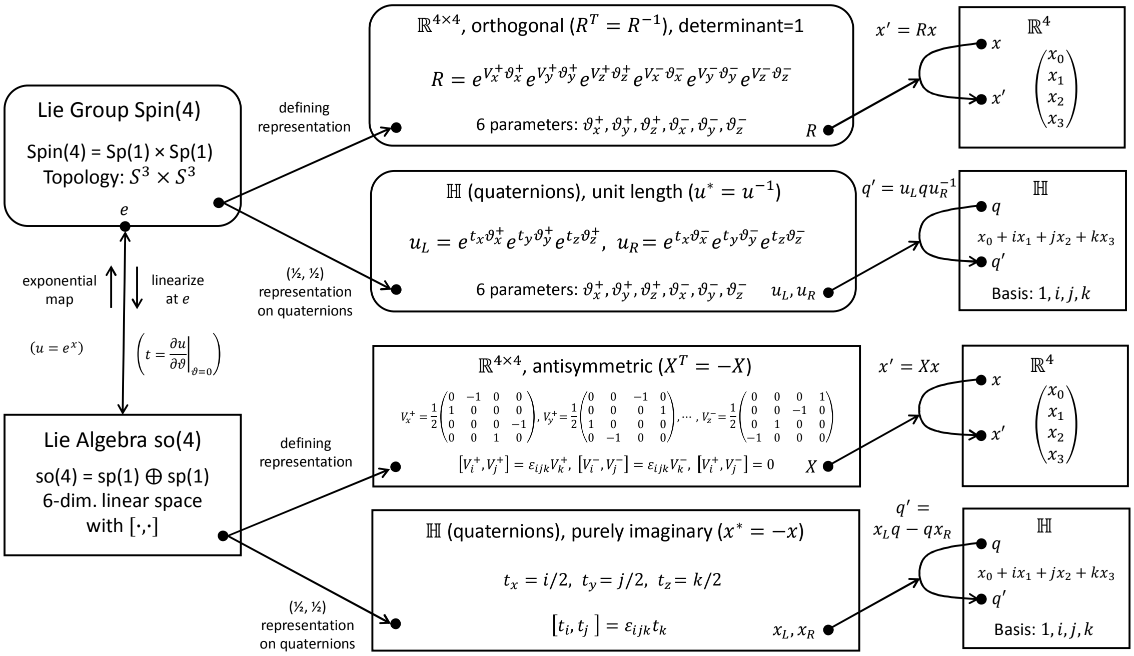
5.10 Spin(4): The (½, ½) Representation; 4D Rotation with Two Quaternions



When we discussed Sp(1) a while back, we found that there is a “dictionary” relating unit-length quaternions to SU(2) matrices, thus establishing the isomorphism Sp(1) = SU(2). Let’s use this “dictionary” to represent Spin(4) by two unit quaternions instead of two SU(2) matrices.

The upper branch of the diagram shows again the defining representation of Spin(4) in which a real 4×4 matrix R acts on a real 4-component vector x, while the lower branch shows the (½, ½) representation of Spin(4) in which two unit quaternions, u_L and u_R, act on a general quaternion $q \in \mathbb{H}$ like $q' = u_L q u_R^{-1}$. This is a direct “translation” of the action $z' = U_L z U_R^\dagger = U_L z U_R^{-1}$ from the previous example. The left unit quaternion depends only on the self-dual double-rotation angles ϑ_i^+ and the right unit quaternion depends only on the anti-self-dual double-rotation angles ϑ_i^-; the transformation as a whole depends on all six parameters. Note that the two unit quaternions are (defining) representations of the two factors in the decomposition Spin(4) = Sp(1) × Sp(1). The Lie algebra consists of pairs of purely imaginary quaternions, x_L and x_R, that act on a general quaternion like $q' = x_L q - q x_R$. This is analogous to the action $z' = X_L z + z X_R^\dagger = X_L z - z X_R$ from the previous example.

The representation space can be understood as a 1-dimensional *quaternionic* vector space or as a 4-dimensional *real* vector space. When interpreted as a 4D real vector space with the basis $1, i, j, k$ (see the diagram) the (½, ½) representation is equivalent to the defining representation of Spin(4).

To check this equivalence with an example, we pick (again) the self-dual double rotation for which $\vartheta_z^+ \neq 0$ and $\vartheta_x^+ = \vartheta_y^+ = \vartheta_x^- = \vartheta_y^- = \vartheta_z^- = 0$. In this case, the unit-quaternion pair is $u_L = \exp(t_z\vartheta_z^+) = \exp(k/2 \cdot \vartheta_z^+) = \cos(\vartheta_z^+/2) + k\sin(\vartheta_z^+/2)$ and $u_R = 1$ and the general quaternion transforms like $q' = [\cos(\vartheta_z^+/2) + k\sin(\vartheta_z^+/2)]q$:

$$x_0' + ix_1' + jx_2' + kx_3' = [\cos(\vartheta_z^+/2) + k\sin(\vartheta_z^+/2)](x_0 + ix_1 + jx_2 + kx_3).$$

Multiplying out the quaternions, simplifying $ki = j, kj = -i, k^2 = -1$, and solving for the x_i', we find that the vector x_i transforms like

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\vartheta_z^+/2 & 0 & 0 & -\sin\vartheta_z^+/2 \\ 0 & \cos\vartheta_z^+/2 & -\sin\vartheta_z^+/2 & 0 \\ 0 & \sin\vartheta_z^+/2 & \cos\vartheta_z^+/2 & 0 \\ \sin\vartheta_z^+/2 & 0 & 0 & \cos\vartheta_z^+/2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

This transformation matrix exactly matches the self-dual double rotation matrix $R_z(\vartheta_z^+)$ from our earlier example! Repeating this exercise for the remaining five double rotations confirms that the two representations are equivalent.

Just to be sure, let's also check the action of some algebra elements. The generator pair for the above double rotation is $x_L = t_z = k/2$ and $x_R = 0$, that is, the quaternion transforms like $q' = (k/2)q$:

$$x_0' + ix_1' + jx_2' + kx_3' = (k/2)(x_0 + ix_1 + jx_2 + kx_3).$$

Solving for the x_i', recovers the basis generator V_z^+ from our earlier example:

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Similarly, the generator pair $x_L = 0$ and $x_R = t_z = k/2$ transforms the quaternion like $q' = -q(k/2)$:

$$x_0' + ix_1' + jx_2' + kx_3' = -(x_0 + ix_1 + jx_2 + kx_3)(k/2).$$

Once more, solving for the x_i', recovers the basis generator V_z^- from our earlier example:

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

What happens if we restrict ourselves to $u_L = u_R = u$? Expanding q in the transformation $q' = uqu^{-1}$, reveals that this kind of restricted 4D transformation leaves x_0 unchanged:

$$x_0' + ix_1' + jx_2' + kx_3' = u(x_0 + ix_1 + jx_2 + kx_3)u^{-1} = x_0 + u(ix_1 + jx_2 + kx_3)u^{-1}.$$

In other words, the transformation describes a *3D rotation* in the subspace x_1, x_2, x_3! This agrees perfectly with what we have found earlier when discussing Sp(1): The 3-dimensional representation of Sp(1), which acts like $q' = uqu^{-1}$ on purely imaginary quaternions ($x_0 = 0$), describes 3D rotations!

For a beautiful visualization of 4D and 3D rotations using quaternions, see https://www.3blue1brown.com/lessons/quaternions and https://eater.net/quaternions.