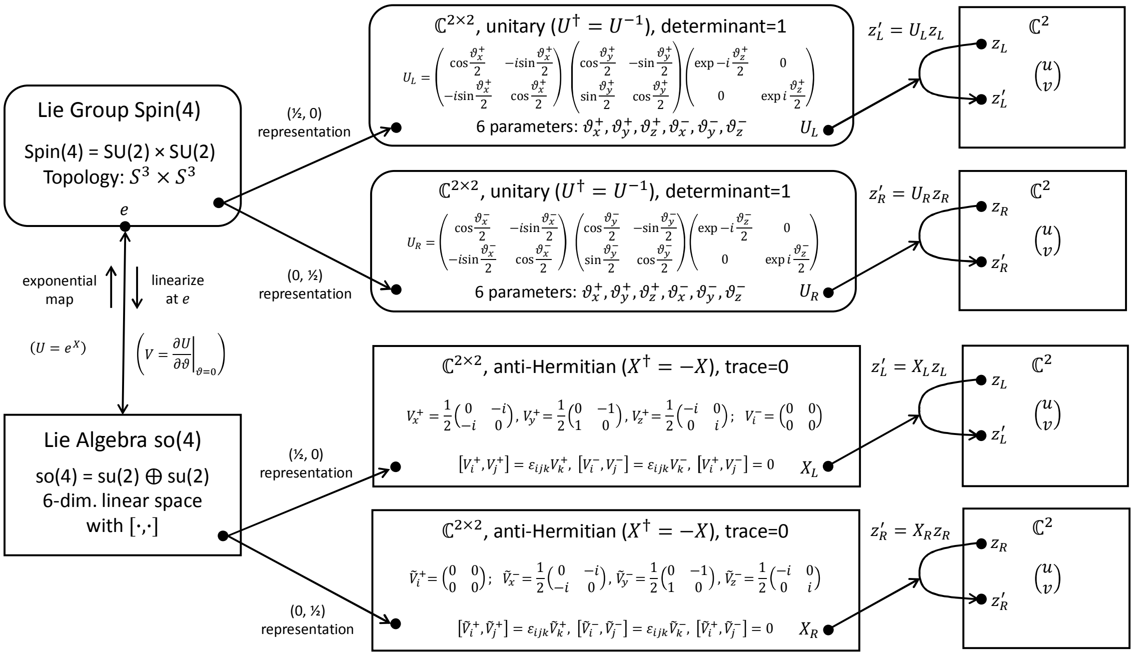
5.7 Spin(4): The (½, 0) and (0, ½) Representations



So far, we have used the 1- and 3-dimensional representations of so(3) to construct various representations of so(4). But we know that so(3) also has a 2-dimensional representation acting on complex 2-vectors known as *spinors*. Now, we are going to use this fact to construct 2-dimensional *spinor representations* of so(4)!

Let's start with the (½, 0) representation of so(4). Remembering that su(2) is isomorphic to so(3), we use the 2-dimensional (defining) representation ($j = ½$) of su(2), for the basis generators V_i^+, that is,

$$V_x^+ = \frac{1}{2}\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix},\; V_y^+ = \frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\; V_z^+ = \frac{1}{2}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix},$$

and the 1-dimensional representation ($j = 0$) for the basis generators V_i^-, that is,

$$V_x^- = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\; V_y^- = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\; V_z^- = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

These matrices satisfy all the commutation relations of so(4), including $[V_i^+, V_j^-] = 0$.

Exponentiating this Lie algebra yields the unitary transformation $U_L = \exp(V_x^+\vartheta_x^+)\cdot\exp(V_y^+\vartheta_y^+)\cdot\exp(V_z^+\vartheta_z^+)$, which we are already familiar with from SU(2). Note that this transformation depends only on the first three parameters, ϑ_x^+, ϑ_y^+, ϑ_z^+; alternatively, if we go back to the plane-rotation parameters, it depends on θ_x, θ_y, θ_z and ϕ_x, ϕ_y, ϕ_z in the *same* way. We have found a 2-dimensional spinor representation of Spin(4) (see the upper branch of the diagram)!

The construction of the (0, ½) representation proceeds along the same lines. The main difference is that the unitary transformation U_R now depends only on the second three parameters ϑ_x^-, ϑ_y^-, ϑ_z^- or, if expressed in terms of plane-rotation parameters, on θ_x, θ_y, θ_z and ϕ_x, ϕ_y, ϕ_z in *opposite* ways. We have found another 2-dimensional spinor representation of Spin(4) (see the lower branch of the diagram)!

Are the (½, 0) and (0, ½) representations equivalent? No, despite looking very similar! It is not possible to turn them into each other with a similarity transformation because the respective matrices depend on two different subsets of the six parameters. Remember that the (1, 0) and (0, 1) representations were also inequivalent. In fact, all $(j, 0)$ and $(0, j)$ representations of Spin(4) are inequivalent, irreducible representations.

Given the (½, 0) and (0, ½) representations, we can easily construct a (reducible) 4-dimensional representation of Spin(4) by taking the direct sum (½, 0) ⊕ (0, ½). Its basis generators are

$$V_x^+ = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ V_y^+ = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ V_z^+ = \frac{1}{2}\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V_x^- = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix},\ V_y^- = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ V_z^- = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

This representation is different from the *irreducible* 4-dimensional representation that defines SO(4). We'll see how to construct the latter representation in a moment.