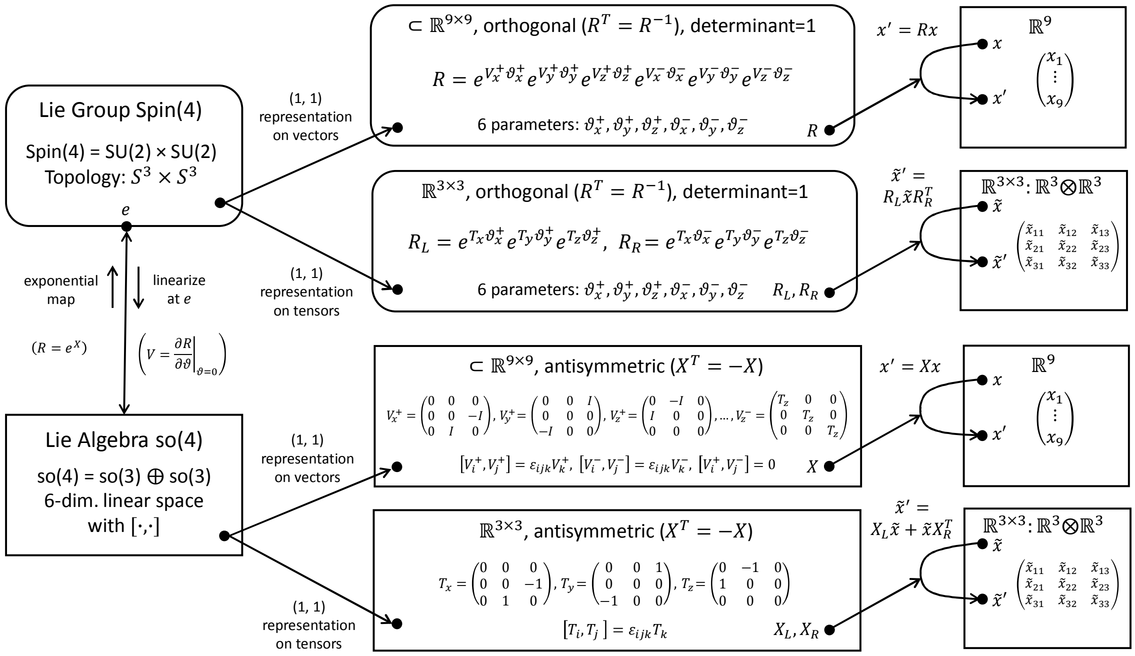
5.8 Spin(4): The (1, 1) Representation; Tensor Product (1, 0) ⊗ (0, 1)



Having explored the (0, 0), (1, 0), and (0, 1) representations, we now want to construct the 9-dimensional (1, 1) representation of Spin(4). We will do this using two slightly different approaches.

One approach is to try and find two sets of matrices, such that each set satisfies the commutation relations of so(3) and the two sets commute among each other. After some trial and error, we find the following six 9×9 basis generators: $V_i^+ = T_i \otimes I$ and $V_i^- = I \otimes T_i$, where ⊗ is the Kronecker product, T_i are the usual three 3×3 basis generators of so(3), and I is the 3×3 identity matrix. Using block-matrix notation, we have

$$V_x^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -I \\ 0 & I & 0 \end{pmatrix},\ V_y^+ = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ -I & 0 & 0 \end{pmatrix},\ V_z^+ = \begin{pmatrix} 0 & -I & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$V_x^- = \begin{pmatrix} T_x & 0 & 0 \\ 0 & T_x & 0 \\ 0 & 0 & T_x \end{pmatrix},\ V_y^- = \begin{pmatrix} T_y & 0 & 0 \\ 0 & T_y & 0 \\ 0 & 0 & T_y \end{pmatrix},\ V_z^- = \begin{pmatrix} T_z & 0 & 0 \\ 0 & T_z & 0 \\ 0 & 0 & T_z \end{pmatrix}.$$

Note how the two so(3) algebras operate at two different levels: the V_i^+ generators make use of the outer (block-matrix) level and the V_i^- generators make use of the inner (submatrix) level. The two sets of matrices commute because the V_i^- "behave" like scalars at the block level. Exponentiating this 9-dimensional representation of the algebra gives us the (1, 1) representation of the group. The upper branch of the diagram illustrates this representation, which acts on the 9-component column vector x.

A more systematic way of constructing the (1, 1) representation is to take the tensor product of the two 3-dimensional representations that we already know: (1, 1) = (1, 0) ⊗ (0, 1). Earlier, we took the tensor product of two *copies* of the same representation. Now, we take the tensor product of two *different* representations, namely (1, 0) and (0, 1). The lower branch of the diagram illustrates this tensor-product representation, which acts on the 3×3 tensor $\tilde{x}$.

We know that Spin(4) = SU(2) × SU(2). This means that each element R of a Spin(4) representation can be expressed as a *pair* of elements, R_L and R_R, drawn from an SU(2) representation. Similarly, because of spin(4) = so(4) = so(3) ⊕ so(3) each element X of an so(4) representation can be expressed as a pair of elements, X_L and X_R, drawn from an so(3) representation. The lower branch of the diagram shows two copies of the 3-dimensional representation of so(3) and two copies of the corresponding (real) SU(2) representation.

How does the tensor $\tilde{x}$ transform under the (1, 1) representation? Consider the outer product (= tensor product) of the 3-component column vector x_L from the (1, 0) representation space and the 3-component column vector x_R from the (0, 1) representation space: $\tilde{x} = x_L x_R^T$. Whereas this is a special tensor (a simple product), it transforms like a general tensor. We know that the (1, 0) representation acts like $x_L' = R_L(\vartheta_x^+, \vartheta_y^+, \vartheta_z^+) x_L$, where R_L is a 3×3 rotation matrix, and that the (0, 1) representation acts like $x_R' = R_R(\vartheta_x^-, \vartheta_y^-, \vartheta_z^-) x_R$, where R_R is another 3×3 rotation matrix. From these facts, we conclude that $\tilde{x}$ transforms like $\tilde{x}' = x_L' x_R'^T = R_L x_L (R_R x_R)^T = R_L x_L x_R^T R_R^T = R_L \tilde{x} R_R^T$. Note that the two rotation matrices are 3-dimensional representations of the two factors in the decomposition Spin(4) = SU(2) × SU(2). In conclusion, each element of Spin(4) maps to two 3×3 rotation matrices, R_L and R_R, which act jointly on the tensor $\tilde{x}$ and collectively depend on six parameters (see the lower branch of the diagram).

To find the six basis generators and their action on the tensor $\tilde{x}$, we take the derivative of the transformation $\tilde{x}' = R_L \tilde{x} R_R^T$ with respect to the six parameters and evaluate the result at $\vartheta_i^+ = \vartheta_i^- = 0$ (which parametrizes the identity transformation). For the three parameters ϑ_i^+ we find $\tilde{x}' = T_i \tilde{x}$ and for the three parameters ϑ_i^- we find $\tilde{x}' = \tilde{x} T_i^T$. Thus, the action of an arbitrary generator is $\tilde{x}' = X_L \tilde{x} + \tilde{x} X_R^T$, where X_L is a 3×3 matrix in the basis T_x, T_y, T_z and X_R is another 3×3 matrix in that same basis. In conclusion, each element of so(4) maps to two (antisymmetric) 3×3 matrices, X_L and X_R, which act jointly on the tensor $\tilde{x}$ (see the lower branch of the diagram).

Now that we understand how the (1, 1) representation acts on 3×3 tensors, $\tilde{x}$, we can go back and ask how the same representation acts on 9-component column vectors, x. Following a similar reasoning as when we "flattened" the tensor product representation of SU(2), we find that the group action $\tilde{x}' = R_L \tilde{x} R_R^T$ becomes $x' = (R_L \otimes R_R) x$ and the algebra action $\tilde{x}' = X_L \tilde{x} + \tilde{x} X_R^T$ becomes $x' = (X_L \otimes I + I \otimes X_R) x$, where ⊗ is again the Kronecker product. In other words, $X = X_L \otimes I + I \otimes X_R$ is the flattened generator and we conclude that $T_i \otimes I$, $I \otimes T_i$ is indeed an appropriate set of basis generators, as we found by trial and error in our first approach.

Have we met this irreducible (1, 1) representation of Spin(4) before? Yes, it turns out to be equivalent to the 9-dimensional traceless symmetric tensor representation of SO(4).