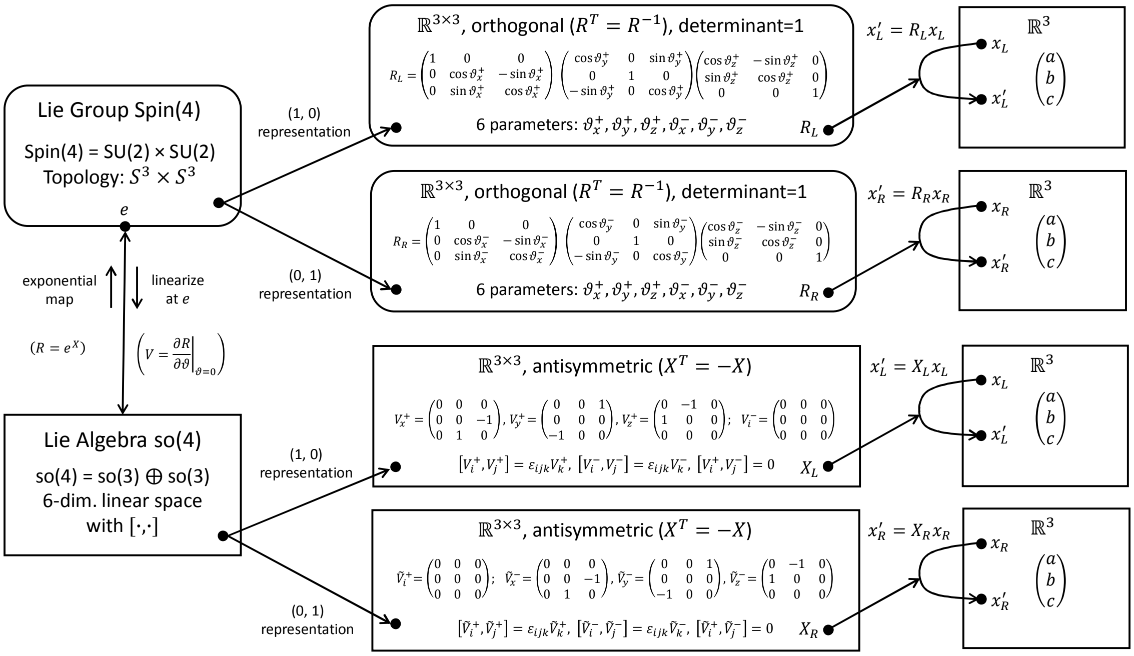
5.6 Spin(4): The (0, 0), (1, 0), and (0, 1) Representations



In the previous example, we saw that the so(4) algebra breaks up into two so(3) algebras: so(4) = so(3) ⊕ so(3). This observation provides us with the key to enumerate and construct all the irreducible representations of so(4)! We know that all the irreducible representations of so(3) can be enumerated by their dimension d or, equivalently, by their spin j, where $d = 2j + 1$. Thus, all irreducible representations of so(4) can be enumerated by *pairs* of dimensions (d_1, d_2) or, equivalently, by pairs of spins (j_1, j_2). In the following, we use the more common spin notation. By putting the irreducible representations of so(4) into the exponential map, we can find the irreducible representations of Spin(4). (Also see the Appendix "The Irreducible Representations of the Lorentz Group".)

Let's construct the (0, 0) representation of so(4). We use the 1-dimensional representation ($j = 0$) of so(3) for the basis generators V_i^+, that is, $V_x^+ = V_y^+ = V_z^+ = 0$, and again for V_i^-, that is, $V_x^- = V_y^- = V_z^- = 0$. Furthermore, we need to check if the two sets of basis generators commute, $[V_i^+, V_j^-] = 0$, which they do because they are scalars. Thus, we have found the basis generators of (0, 0). What does this representation do? It is the *trivial representation*, which leaves everything unchanged!

Next, let's tackle the (1, 0) representation. Now, we use the 3-dimensional representation ($j = 1$) of so(3) for the basis generators V_i^+, that is,

$$V_x^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ V_y^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ V_z^+ = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the 1-dimensional representation ($j = 0$) of so(3) for the basis generators V_i^-. Since all generators need to act on 3D vectors, we upgrade the 1-dimensional basis generators to 3×3 matrices, that is,

$$V_x^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ V_y^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ V_z^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Checking $[V_i^+, V_j^-] = 0$, we find that we are good. A general element, X_L, of this algebra is a linear combination of these six basis generators, where the last three generators make no contribution: $X_L = \alpha V_x^+ + \beta V_y^+ + \gamma V_z^+$. See the upper branch of the diagram.

What does this 3-dimensional representation do? Exponentiating the Lie algebra, we find the transformation $R_L = \exp(V_x^+ \vartheta_x^+) \cdot \exp(V_y^+ \vartheta_y^+) \cdot \exp(V_z^+ \vartheta_z^+)$, where the first three parameters, ϑ_x^+, ϑ_y^+, ϑ_z^+, control ordinary 3D rotations and the remaining three parameters, ϑ_x^-, ϑ_y^-, ϑ_z^-, have no effect. Alternatively, if we go back to the plane-rotation parameters, we find that θ_x, θ_y, θ_z and ϕ_x, ϕ_y, ϕ_z both control the *same* ordinary 3D rotations. We have encountered this 3-dimensional representation before: it is the *self-dual representation* of SO(4)!

The construction of the (0, 1) representation proceeds along the same lines (see the lower branch of the diagram). The basis generators are

$$V_x^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; V_y^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; V_z^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$V_x^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, V_y^- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, V_z^- = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The transformation matrix is $R_R = \exp(V_x^- \vartheta_x^-) \cdot \exp(V_y^- \vartheta_y^-) \cdot \exp(V_z^- \vartheta_z^-)$. Now, the first three parameters, ϑ_x^+, ϑ_y^+, ϑ_z^+, have no effect and the second three parameters, ϑ_x^-, ϑ_y^-, ϑ_z^-, control ordinary 3D rotations. In terms of plane-rotation parameters, we find that θ_x, θ_y, θ_z and ϕ_x, ϕ_y, ϕ_z control ordinary 3D rotations by equal amounts but in *opposite* directions. Not surprisingly, the (0, 1) representation is the *anti-self-dual representation* of SO(4) that we discussed earlier.

We used the indices L and R for left and right (e.g., in R_L and R_R or X_L and X_R) to indicate whether we are dealing with a $(j, 0)$ or a $(0, j)$ representation, respectively. The advantage of this notation will become apparent in the subsequent examples.

Given the (1, 0) and (0, 1) representations, we can easily construct a (reducible) 6-dimensional representation by taking the direct sum (1, 0) ⊕ (0, 1). Its basis generators are

$$V_x^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, V_y^+ = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, V_z^+ = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V_x^- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, V_y^- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, V_z^- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It turns out that this representation is equivalent to the antisymmetric tensor representation of SO(4), which was our starting point for finding the self-dual and anti-self-dual representations. (Incidentally, it is also equivalent to the adjoint representation of SO(4).)