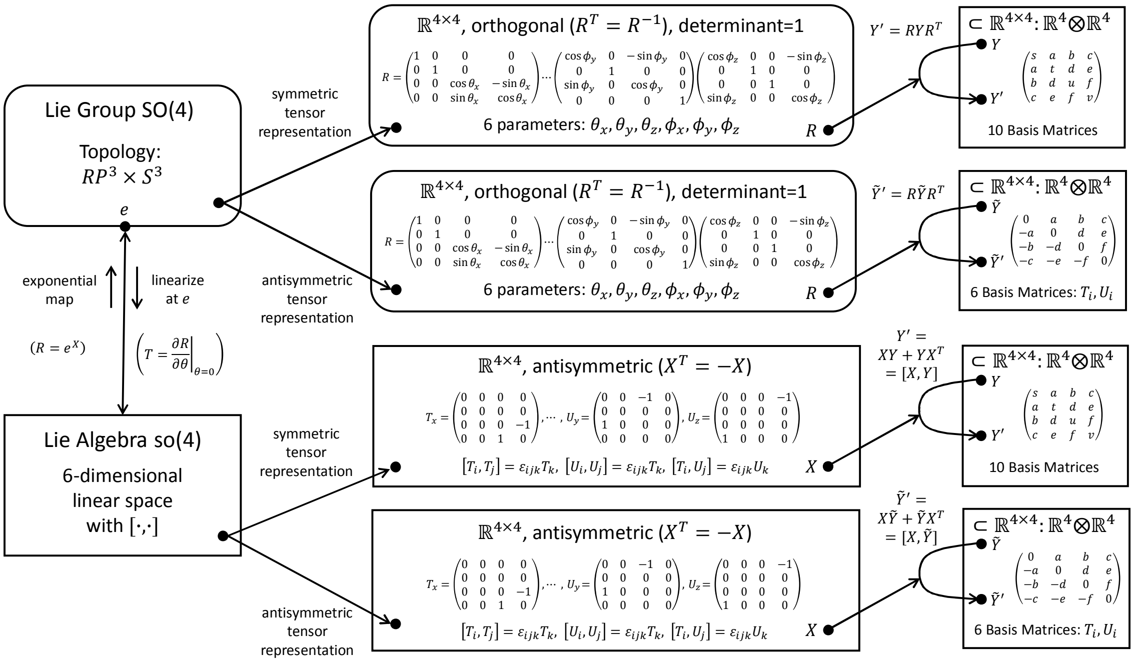
## 5.2 SO(4): Symmetric and Antisymmetric Tensor Representations




Given the defining representation, we can construct a tensor-product representation from two copies of it. We have done this before for SU(2) and SO(3). In the case of SO(4), the tensor-product representation acts on 4×4 matrices and thus is sixteen dimensional. We know from SO(3) that the tensor-product representation is reducible: $\mathbf{3}\otimes\mathbf{3} = \mathbf{5}\oplus\mathbf{3}\oplus\mathbf{1}$. In the case of SO(4), the tensor product can be broken up into a 10-dimensional symmetric and a 6-dimensional antisymmetric representation. Moreover, the symmetric part can be broken up into a 9-dimensional traceless symmetric representation and a 1-dimensional (trivial) representation for the trace. Thus, we can write $\mathbf{4}\otimes\mathbf{4} = \mathbf{9}\oplus\mathbf{6}\oplus\mathbf{1}$.

The 10-dimensional symmetric and the 6-dimensional antisymmetric representation are illustrated in the upper and lower branch of the diagram, respectively.

From SU(2) and SO(3), we know how the elements of the defining representation act on the tensor-product space. The defining matrix, $R$, of the group acts on the matrix, $Y$, in the representation space according to $Y' = RYR^T$ and the defining matrix, $X$, of the algebra acts on the matrix in the representation space according to $Y' = XY + YX^T$. Because $R$ is orthogonal ($R^T = R^{-1}$), we can also write the group action as $Y' = RYR^{-1}$ and because $X$ is antisymmetric ($X^T = -X$), we can also write the algebra action as $Y' = [X, Y]$. Thus, we see that the antisymmetric tensor representation and the adjoint representation (from the previous example) are the same thing!

What distinguishes a symmetric from an antisymmetric matrix? A symmetric matrix remains unchanged when being transposed: $Y^T = Y$, we might as well call it the *self-transpose*. In contrast, an antisymmetric matrix changes its sign when being transposed: $\tilde{Y}^T = -\tilde{Y}$, deserving the name *anti-self-transpose*. This is analogous to the concepts of self-dual and anti-self-dual, which we will encounter in the next example.

Let's try to understand why the symmetric and antisymmetric tensors furnish separate representations. Geometrically, this means that the symmetric and antisymmetric parts of a tensor must represent two different objects. Indeed, we can interpret the symmetric tensor $Y$ as the quadratic hypersurface defined by the points (vectors) $x$ that satisfy $x^T Y x = 1$. For the tensor components shown in the upper branch of the diagram, this expands to $sw^2 + tx^2 + uy^2 + vz^2 + 2awx + 2bwy + 2cwz + 2dxy + 2exz + 2fyz = 1$. Moreover, we can interpret an antisymmetric tensor as a (generalized) oriented area element as follows. Given the antisymmetric tensor $\tilde{Y}$ (which has six free parameters in the 4D case), we can find two 4-component column vectors $x$ and $y$, such that $\tilde{Y} = xy^T - yx^T$ (known as the wedge product $\tilde{Y} = x \wedge y$). Then, $\tilde{Y}$ describes the area of the parallelogram enclosed by the two vectors along with its orientation in the ambient 4D space. When performing a rotation, the hypersurface and the area element transform as two separate objects.

Formally, the fact that symmetric and antisymmetric tensors furnish independent representations means that if $Y^T = Y$ holds, then so does $Y'^T = Y'$ and if $\tilde{Y}^T = -\tilde{Y}$ holds, then so does $\tilde{Y}'^T = -\tilde{Y}'$, in other words, symmetric and antisymmetric tensors don't mix under transformation. These if-then implications are easy to verify. The conclusion of the first statement, $Y'^T = Y'$, can be expanded to $(RYR^T)^T = RYR^T$. Applying the transpose-operator rule $(XY)^T = Y^T X^T$ to the left-hand side of the equality, we get $R(RY)^T = RYR^T$ and applying it once more, we get $RY^T R^T = RYR^T$. Thus, given the premise, $Y^T = Y$, the conclusion is indeed true. Similarly, the conclusion of the second statement, $\tilde{Y}'^T = -\tilde{Y}'$, can be expanded to $(R\tilde{Y}R^T)^T = -R\tilde{Y}R^T$. Switching the transpose operators like before, we get $R\tilde{Y}^T R^T = -R\tilde{Y}R^T$. Once again, given the premise, $\tilde{Y}^T = -\tilde{Y}$, the conclusion turns out to be true. Incidentally, the above reasoning did not make use of the orthogonality or the unit determinant of the matrix $R$. Thus, the conclusion does not only apply to tensor-product representations of SO(n) but more generally to those of GL(n), that is, the group of invertible linear transformations (G = general, L = linear).

So far, SO(4) appears to behave qualitatively just like SO(3). But here is something new: In the case of SO(3), $\mathbf{3} \otimes \mathbf{3} = \mathbf{5} \oplus \mathbf{3} \oplus \mathbf{1}$ was the full decomposition into irreducible representations; in the case of SO(4), however, the 6-dimensional antisymmetric tensor representation can be broken up once more into two 3-dimensional representations known as the *self-dual* and *anti-self-dual* tensor representations! Thus, the full decomposition into irreducible representations is $\mathbf{4} \otimes \mathbf{4} = \mathbf{9} \oplus \mathbf{3} \oplus \overline{\mathbf{3}} \oplus \mathbf{1}$.