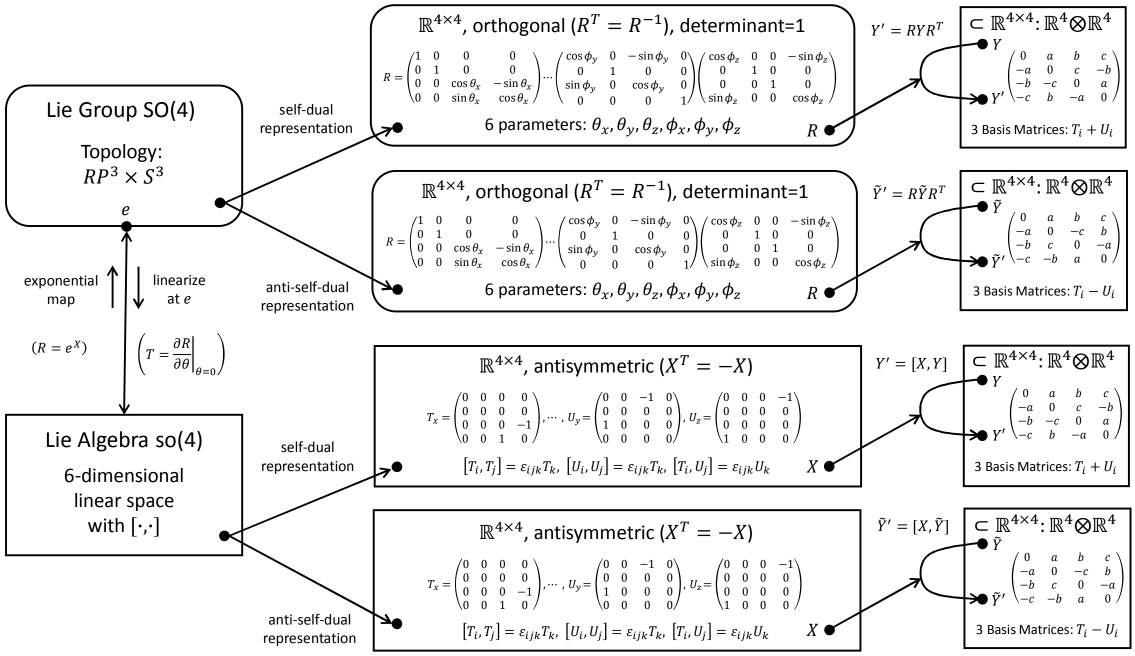
5.3 SO(4): Self-Dual and Anti-Self-Dual Tensor Representations



Before we can discuss the self-dual and anti-self-dual tensor representations, we need to introduce the *Hodge dual* (a.k.a. *Hodge star operation*). Despite the name, the Hodge dual is *not* related to dual vectors (= covectors). The Hodge dual can only be taken of a totally antisymmetric tensor and results again in a totally antisymmetric tensor. Specifically, for a 4-dimensional Euclidean space the Hodge dual of the antisymmetric rank-2 tensor Y_{kl} is defined to be the antisymmetric rank-2 tensor $\hat{Y}_{ij} = \frac{1}{2}\varepsilon_{ijkl}Y_{kl}$, where a summation over the repeated indices k and l is implied [GTNut, Ch. IV.1, p. 197].

The 4-dimensional (rank-4) Levi-Civita symbol ε_{ijkl} is defined to be $+1$ if its indices are an even permutation of 1234, -1 if its indices are an odd permutation of 1234, and 0 in all other cases. Thus, the nonzero elements of ε_{ijkl} (sorted by the first two indices) are:

$\varepsilon_{1234} = 1$	$\varepsilon_{3124} = 1$	$\varepsilon_{1243} = -1$	$\varepsilon_{3142} = -1$
$\varepsilon_{1342} = 1$	$\varepsilon_{3241} = 1$	$\varepsilon_{1324} = -1$	$\varepsilon_{3214} = -1$
$\varepsilon_{1423} = 1$	$\varepsilon_{3412} = 1$	$\varepsilon_{1432} = -1$	$\varepsilon_{3421} = -1$
$\varepsilon_{2143} = 1$	$\varepsilon_{4132} = 1$	$\varepsilon_{2134} = -1$	$\varepsilon_{4123} = -1$
$\varepsilon_{2314} = 1$	$\varepsilon_{4213} = 1$	$\varepsilon_{2341} = -1$	$\varepsilon_{4231} = -1$
$\varepsilon_{2431} = 1$	$\varepsilon_{4321} = 1$	$\varepsilon_{2413} = -1$	$\varepsilon_{4312} = -1$

Evaluating $\hat{Y}_{ij} = \frac{1}{2}\varepsilon_{ijkl}Y_{kl}$ for the general antisymmetric 4×4 matrix shown on the left side, we find the dual matrix shown to the right side:

$$Y = \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & \nu & \mu \\ -\beta & -\nu & 0 & \lambda \\ -\gamma & -\mu & -\lambda & 0 \end{pmatrix} \rightarrow \hat{Y} = \begin{pmatrix} 0 & \lambda & -\mu & \nu \\ -\lambda & 0 & \gamma & -\beta \\ \mu & -\gamma & 0 & \alpha \\ -\nu & \beta & -\alpha & 0 \end{pmatrix}.$$

The dual matrix is again antisymmetric, as expected, but its components got moved around. If we apply the Hodge dual operation for a second time, we get back to the original matrix. In that sense, taking the Hodge dual is analogous to taking the transpose. (For a broader discussion of the Hodge dual, see the Appendix "The Hodge Dual in Euclidean Space".)

Now, if a matrix remains unchanged when taking the Hodge dual, it is called *self-dual* and if a matrix changes its sign when taking the Hodge dual, it is called *anti-self-dual*. This is completely analogous to what we said in the previous example about taking the transpose and the definition of symmetric and antisymmetric matrices. Given any antisymmetric matrix Y, we can decompose it into the self-dual matrix $Y^+ = \frac{1}{2}(Y + \hat{Y})$ and the anti-self-dual matrix $Y^- = \frac{1}{2}(Y - \hat{Y})$ such that $Y = Y^+ + Y^-$. Using the component names from before, we find

$$Y^+ = \frac{1}{2}\begin{pmatrix} 0 & \alpha+\lambda & \beta-\mu & \gamma+\nu \\ -\alpha-\lambda & 0 & \nu+\gamma & \mu-\beta \\ -\beta+\mu & -\nu-\gamma & 0 & \lambda+\alpha \\ -\gamma-\nu & -\mu+\beta & -\lambda-\alpha & 0 \end{pmatrix}, \quad Y^- = \frac{1}{2}\begin{pmatrix} 0 & \alpha-\lambda & \beta+\mu & \gamma-\nu \\ -\alpha+\lambda & 0 & \nu-\gamma & \mu+\beta \\ -\beta-\mu & -\nu+\gamma & 0 & \lambda-\alpha \\ -\gamma+\nu & -\mu-\beta & -\lambda+\alpha & 0 \end{pmatrix}.$$

Note that each of those matrices has only *three* independent parameters, shown in red, green, and blue:

$$Y^+ = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{pmatrix}, \quad Y^- = \begin{pmatrix} 0 & a' & b' & c' \\ -a' & 0 & -c' & b' \\ -b' & c' & 0 & -a' \\ -c' & -b' & a' & 0 \end{pmatrix}.$$

In fact, the self-dual matrix Y^+ can be expressed in the 3D basis $T_x + U_x$, $T_y + U_y$, and $T_z + U_z$ and the anti-self-dual matrix Y^- can be expressed in the 3D basis $T_x - U_x$, $T_y - U_y$, and $T_z - U_z$.

The diagram shows the self-dual and anti-self-dual representations of SO(4). In the upper branch, the 4D rotations act on self-dual matrices, which can be expressed in the 3D basis $T_i + U_i$, and in the lower branch, the same rotations act on anti-self-dual matrices, which can be expressed in the 3D basis $T_i - U_i$. The 4D rotations always map self-dual matrices to self-dual ones and anti-self-dual matrices to anti-self-dual ones, never mixing the two together [GTNut, IV.1, p. 197].

Let's try this out by acting with the generator T_x on the general self-dual matrix Y^+. (The calculation is easier with a generator than a rotation matrix.) In other words, we calculate $[T_x, Y^+]$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{pmatrix} - \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -c & b \\ 0 & 0 & b & c \\ c & -b & 0 & 0 \\ -b & -c & 0 & 0 \end{pmatrix}.$$

Indeed, the result is again a self-dual matrix, albeit one that is less general than the one we started with. Note that the three independent parameters (a, b, c) in the original self-dual matrix get mapped to $(0, -c, b)$, a fact that we will return to in the next example. We can do the same calculation with the remaining five basis generators and find that the result is always a self-dual matrix. Finally, we can let all the basis generators act on a general anti-self-dual matrix and find that the result is always an anti-self-dual matrix.