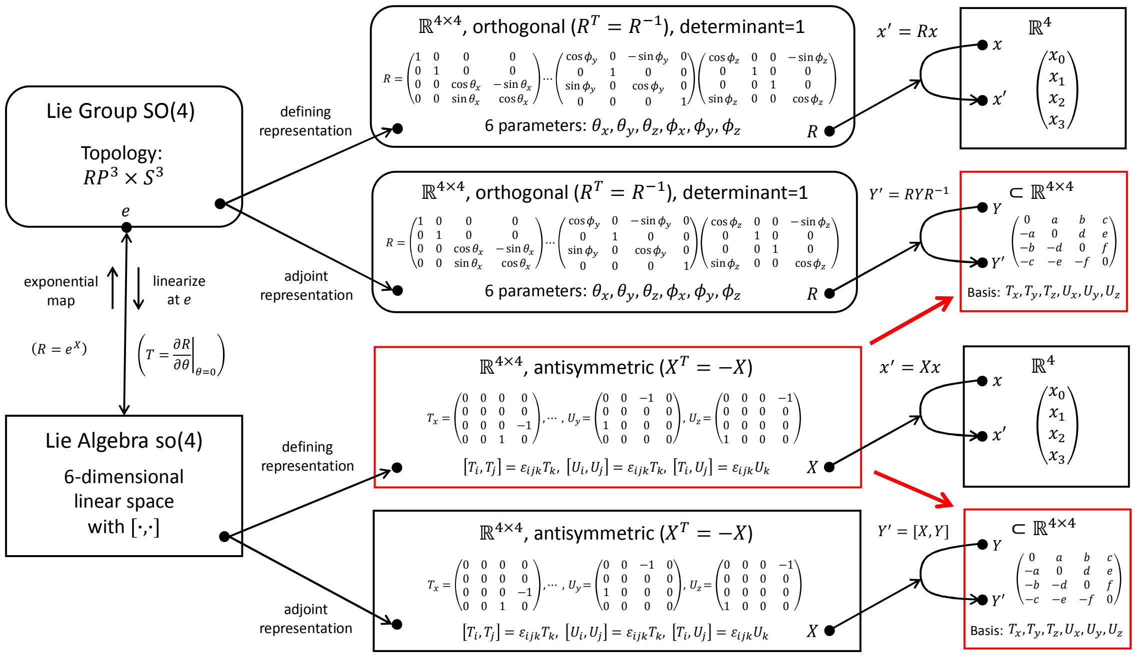
5. Rotation in Four Dimensions and Relativistic Space-Time

5.1 SO(4): The Group of Rotations in 4-Dimensional Euclidean Space



We now turn to rotation in 4-dimensional Euclidean space. This is a good warm-up exercise to get ready for the Lorentz transformation, which is at the core of the theory of relativity. Once we understand 4D Euclidean space, it's just a hop and a skip (actually, a flip of sign) to Minkowski space-time!

The crucial thing to understand about spaces with other-than-three dimensions is that rotations are defined by 2D planes, *not* by axes. In 2D space there is only one possible plane and we rotate about a point in this plane. In 3D space there are three possible orthogonal planes and we rotate about an axis, which singles out a particular (orthogonal) plane. In 4D space there are six possible orthogonal planes and we rotate about a plane, which singles out another (orthogonal) plane! Therefore, we need *six rotation angles* to describe a rotation in 4D space. In the following, we write vectors in 4D Euclidean space as $x = (x_0, x_1, x_2, x_3)^T = (w, x, y, z)^T$ without an arrow on top of x (trusting that x as a vector and x as a component can be distinguished from context). The six orthogonal planes can then be identified as yz, zx, xy, wx, wy, and wz. (The reason for numbering the components from 0 to 3 instead of from 1 to 4 is that it will make the transition from 4D space to space-time smoother.)

The rotations in the yz, zx, and xy planes correspond directly to those that we already discussed for SO(3). For the associated rotation angles, we reuse the symbols θ_x, θ_y, and θ_z. For the three new rotations in the wx, wy, and wz planes, we introduce the angles ϕ_x, ϕ_y, and ϕ_z, respectively. A general 4D rotation can be written as the matrix product $R(\theta_x, \theta_y, \theta_z, \phi_x, \phi_y, \phi_z) = R_{yz}(\theta_x) \cdot R_{zx}(\theta_y) \cdot R_{xy}(\theta_z) \cdot R_{wx}(\phi_x) \cdot R_{wy}(\phi_y) \cdot R_{wz}(\phi_z)$, where

$$R_{yz} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_x & -\sin\theta_x \\ 0 & 0 & \sin\theta_x & \cos\theta_x \end{pmatrix},\ R_{zx} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_y & 0 & \sin\theta_y \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix},\ R_{xy} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & -\sin\theta_z & 0 \\ 0 & \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_{wx} = \begin{pmatrix} \cos\phi_x & -\sin\phi_x & 0 & 0 \\ \sin\phi_x & \cos\phi_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; R_{wy} = \begin{pmatrix} \cos\phi_y & 0 & -\sin\phi_y & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi_y & 0 & \cos\phi_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; R_{wz} = \begin{pmatrix} \cos\phi_z & 0 & 0 & -\sin\phi_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\phi_z & 0 & 0 & \cos\phi_z \end{pmatrix}.$$

The six basis generators are obtained, as usual, by taking the derivatives with respect to the six parameters and evaluating the result at $\theta_i = \phi_i = 0$. For the first three basis generators, we reuse the symbols T_x, T_y, and T_z from SO(3), although they are now 4×4 instead of 3×3 matrices, and for the three new ones, we use the symbols U_x, U_y, and U_z:

$$T_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; T_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; T_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$U_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; U_y = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; U_z = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

A general element, X, of the so(4) Lie algebra is an antisymmetric 4×4 matrix. Such a matrix has six independent parameters, confirming the six degrees of freedom provided by the six rotation angles.

The commutation relations among T_x, T_y, and T_z are the same ones that we already discussed for SO(3), namely $[T_x, T_y] = T_z$, $[T_y, T_z] = T_x$, and $[T_z, T_x] = T_y$, or, written more compactly, $[T_i, T_j] = \varepsilon_{ijk} T_k$, where a summation over the repeated index k is implied. How do the Us commute with each other and with the Ts? It turns out that $[U_i, U_j] = \varepsilon_{ijk} T_k$ and $[T_i, U_j] = \varepsilon_{ijk} U_k$ (see the upper branch of the diagram) [GTNut, Ch. I.3, Appendix 2]. Interestingly, the Ts and Us behave in a similar, but not perfectly identical way! We'll come back to this later.

So far, we discussed the defining, 4-dimensional representation of SO(4), which is shown in the upper branch of the diagram. It is easy to obtain a 6-dimensional representation of SO(4) by constructing the adjoint representation, which is shown in the lower branch. As usual, the adjoint representation acts on the Lie algebra (red arrows). We can think of the adjoint representation as acting on antisymmetric 4×4 matrices by conjugation, $Y' = RYR^{-1}$, or we can think of it as acting on 6-component column vectors (the "unpacked" antisymmetric matrices) by regular matrix-vector multiplication. The diagram illustrates the former case. We have examined these two views of the adjoint representation previously when discussing SU(2).

Similarly, we can think of the adjoint representation of the so(4) algebra as acting on antisymmetric 4×4 matrices by commutation, $Y' = [X, Y]$, or we can think of it as acting on 6-component column vectors by regular matrix-vector multiplication. Again, the diagram illustrates the former case.