## 9.7 The Hodge Dual in Euclidean Space

| | Rank-0 Tensor (Scalar) | Rank-1 Tensor (Vector) | Antisymmetric Rank-2 Tensor | Totally Antisymmetric Rank-3 Tensor | Totally Antisymmetric Rank-4 Tensor |
|---|---|---|---|---|---|
| 2 D | $a$ | $\begin{pmatrix} b \\ c \end{pmatrix}$ | $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ | | |
| 3 D | $a$ | $\begin{pmatrix} b \\ c \\ d \end{pmatrix}$ | $\begin{pmatrix} 0 & d & -c \\ -d & 0 & b \\ c & -b & 0 \end{pmatrix}$ | $\begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}$ | |
| 4 D | $a$ | $\begin{pmatrix} b \\ c \\ d \\ e \end{pmatrix}$ | $\begin{pmatrix} 0 & f & g & h \\ -f & 0 & i & j \\ -g & -i & 0 & k \\ -h & -j & -k & 0 \end{pmatrix}$ | $\begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e & -d \\ 0 & -e & 0 & c \\ 0 & d & -c & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & -e & d \\ 0 & 0 & 0 & 0 \\ e & 0 & 0 & -b \\ -d & 0 & b & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & e & 0 & -c \\ -e & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ c & -b & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -d & c & 0 \\ d & 0 & -b & 0 \\ -c & b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix}$ | $\begin{pmatrix} \mathbf{0} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a \\ 0 & 0 & a & 0 \end{pmatrix} & \mathbf{0} & \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix} & \mathbf{0} & \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \mathbf{0} \end{pmatrix}$ |

Antisymmetric tensors play an important role in mathematics, geometry, and physics. They occur, for example, when taking the *exterior product* of $p$ vectors, which results in a so-called *p-blade* that represents the (oriented) $p$-dimensional volume enclosed by these vectors. (The most general antisymmetric tensor is a *sum* of such $p$-blades.) See the Appendix "The Exterior Product; Area and Volume Elements" for more details. In the following, we look at antisymmetric tensors of various ranks in two, three, and four-dimensional spaces, which leads us to the idea of the *Hodge dual*.

**2D Space** (first row of the diagram). A tensor of rank zero is just a scalar. A 2-dimensional tensor of rank one is just a vector with two independent components: $b$ and $c$. A 2-dimensional tensor of rank two can be written as a 2×2 matrix with four components. If we impose the condition that the tensor must be *antisymmetric*, the two components on the diagonal are forced to zero and the two off-diagonal components must be equal and opposite. Thus, this antisymmetric tensor has only one independent component: $a$. In fact, this tensor can be written as $a\varepsilon_{ij}$, where $\varepsilon_{ij}$ is the 2-dimensional (rank-2) Levi-Civita symbol. A 2-dimensional tensor of rank three has 2×2×2=8 components, but if we impose the condition that it must be (totally) antisymmetric, all components are forced to zero. The same is true for all tensors of higher rank. We notice a pattern in the number of independent components of antisymmetric tensors as we climb up through the ranks: 1 – 2 – 1 followed by all zeros.

**3D Space** (second row of the diagram). A 3-dimensional rank-1 tensor is a vector with three independent components: $b, c$ and $d$. A 3-dimensional rank-2 tensor can be written as a matrix with nine components. If we impose antisymmetry, the number of independent components reduces to three: $b, c$ and $d$. A 3-dimensional rank-3 tensor has 3×3×3=27 components, a kind of 3×3×3 cubic number scheme, best written down as a column of three 3×3 matrices. Now, antisymmetry can occur between any two of the three dimensions of the cube. If we impose antisymmetry with respect to *all* pairs of dimensions, that is, *total* antisymmetry, then only one independent component survives: $a$ (see far right of second row). In fact, this tensor can be written as $a\varepsilon_{ijk}$, where $\varepsilon_{ijk}$ is the 3-dimensional (rank-3) Levi-Civita symbol. (Unfortunately, total antisymmetry is not easily discernable in the column-of-matrices notation.) Totally antisymmetric tensors of rank four and higher are all zero. Again, we notice an intriguing pattern in the number of independent parameters: 1 – 3 – 3 – 1.

**4D Space** (third row of the diagram). By now, you can guess how this pattern continues! Nonzero totally antisymmetric tensors in four dimensions exist up to rank four, thereafter they are all zero. A 4-dimensional rank-4 tensor, a kind of 4×4×4×4 hypercubic number scheme, is best written down as a matrix of matrices. Whereas a general such tensor has 4×4×4×4=256 components, the totally antisymmetric version has only one independent component: $a$. In fact, this tensor is $a\varepsilon_{ijkl}$, where $\varepsilon_{ijkl}$ is the 4-dimensional (rank-4) Levi-Civita symbol. Now, the pattern of independent components goes like: 1 – 4 – 6 – 4 – 1.

Why do totally antisymmetric tensors top out when their rank $p$ equals the dimension $n$ of the space? Because they represent $p$-dimensional volume elements that live *in* the $n$-dimensional space!

**Hodge Dual.** There is a correspondence between totally antisymmetric tensors of rank $p$ and rank $n-p$. For starters, they have the same number of independent components. These tensors, which in the diagram are shown in matching colors (within a row), are Hodge duals of each other! Specifically, for an $n$-dimensional Euclidean space, the *Hodge dual* (a.k.a. *Hodge star operator*) relates a totally antisymmetric tensor of rank $p$, written as $T_{i_1\cdots i_p}$, to one of rank $n-p$, written as $\hat{T}_{i_1\cdots i_{n-p}}$, by

$$\hat{T}_{i_1\cdots i_{n-p}} = \frac{1}{p!}\sum_{j_1\cdots j_p} \varepsilon_{i_1\cdots i_{n-p}j_1\cdots j_p} T_{j_1\cdots j_p}\ ,$$

where $\varepsilon$ is the $n$-dimensional (rank-$n$) Levi-Civita symbol. This operation is abbreviated as $\hat{T} = \star T$. Taking the Hodge dual twice yields the original tensor, except for a sign change if $p(n-p)$ is odd: $\star\star\, T = (-1)^{p(n-p)}\, T$. (In odd-dimensional spaces: $\star\star\, T = T$.) Geometrically, the Hodge dual relates volume elements to their *orthogonal complement* [RtR, Ch. 19.2, p. 444]. For example, in 3D space, it relates an area element to its orthogonal line element.

Let's try the above formula out for $n = 2$. The Hodge dual of the scalar $a$ is the antisymmetric rank-2 tensor $T_{ij} = \varepsilon_{ij}a$, that is, $T_{12} = -T_{21} = a$ and $T_{11} = T_{22} = 0$. Taking the Hodge dual of this tensor again, brings us back to the original scalar $a = \frac{1}{2}\varepsilon_{ij}T_{ij} = \frac{1}{2}(T_{12} - T_{21})$ (summation over repeated indices implied). If $T_{ij}$ is a 2-blade, its Hodge dual $a$ is the (signed) area of the parallelogram enclosed by the vectors making up the blade. The Hodge dual of the vector $v_i$ is the orthogonal vector $u_i = \varepsilon_{ij}v_j$, that is, $(v_2, -v_1)^T$. Taking the Hodge dual of this vector again, yields minus the original vector.

In three dimensions ($n = 3$), the Hodge dual of the antisymmetric rank-2 tensor $T_{ij}$ is the vector $v_i = \frac{1}{2}\varepsilon_{ijk}T_{jk}$. If $T_{ij}$ is a 2-blade, this vector is the *cross product* of the two vectors making up the blade: $\star\left(\vec{a}\wedge\vec{b}\right) = \vec{a}\times\vec{b}$. The Hodge dual of the antisymmetric rank-3 tensor $T_{ijk}$ is the scalar $a = \frac{1}{6}\varepsilon_{ijk}T_{ijk}$. If $T_{ijk}$ is a 3-blade, this scalar is the *triple product* of the three vectors: $\star\left(\vec{a}\wedge\vec{b}\wedge\vec{c}\right) = (\vec{a}\times\vec{b})\cdot\vec{c}$.

In four dimensions ($n = 4$), the Hodge dual of the antisymmetric rank-2 tensor $T_{ij}$ is the tensor $\hat{T}_{ij} = \frac{1}{2}\varepsilon_{ijkl}T_{kl}$, which is again an antisymmetric rank-2 tensor. This "coincidence" leads to the concepts of *self-dual* tensors ($T = \star\, T$) and *anti-self-dual* tensors ($T = -\star\, T$) in 4-dimensional space.