9.6 The Exterior Product; Area and Volume Elements

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\wedge\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}a_1b_1&a_1b_2&a_1b_3\\a_2b_1&a_2b_2&a_2b_3\\a_3b_1&a_3b_2&a_3b_3\end{pmatrix}-\begin{pmatrix}a_1b_1&a_2b_1&a_3b_1\\a_1b_2&a_2b_2&a_3b_2\\a_1b_3&a_2b_3&a_3b_3\end{pmatrix}=\begin{pmatrix}0&a_1b_2-a_2b_1&a_1b_3-a_3b_1\\a_2b_1-a_1b_2&0&a_2b_3-a_3b_2\\a_3b_1-a_1b_3&a_3b_2-a_2b_3&0\end{pmatrix}$$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\wedge\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\wedge\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}a_1b_1c_1&a_1b_1c_2&a_1b_1c_3\\a_1b_2c_1&a_1b_2c_2&a_1b_2c_3\\a_1b_3c_1&a_1b_3c_2&a_1b_3c_3\end{pmatrix}\\\begin{pmatrix}a_2b_1c_1&a_2b_1c_2&a_2b_1c_3\\a_2b_2c_1&a_2b_2c_2&a_2b_2c_3\\a_2b_3c_1&a_2b_3c_2&a_2b_3c_3\end{pmatrix}\\\begin{pmatrix}a_3b_1c_1&a_3b_1c_2&a_3b_1c_3\\a_3b_2c_1&a_3b_2c_2&a_3b_2c_3\\a_3b_3c_1&a_3b_3c_2&a_3b_3c_3\end{pmatrix}\end{pmatrix}+\begin{pmatrix}\begin{pmatrix}a_1b_1c_1&a_1b_2c_1&a_1b_3c_1\\a_2b_1c_1&a_2b_2c_1&a_2b_3c_1\\a_3b_1c_1&a_3b_2c_1&a_3b_3c_1\end{pmatrix}\\\begin{pmatrix}a_1b_1c_2&a_1b_2c_2&a_1b_3c_2\\a_2b_1c_2&a_2b_2c_2&a_2b_3c_2\\a_3b_1c_2&a_3b_2c_2&a_3b_3c_2\end{pmatrix}\\\begin{pmatrix}a_1b_1c_3&a_1b_2c_3&a_1b_3c_3\\a_2b_1c_3&a_2b_2c_3&a_2b_3c_3\\a_3b_1c_3&a_3b_2c_3&a_3b_3c_3\end{pmatrix}\end{pmatrix}+\begin{pmatrix}\begin{pmatrix}a_1b_1c_1&a_2b_1c_1&a_3b_1c_1\\a_1b_1c_2&a_2b_1c_2&a_3b_1c_2\\a_1b_1c_3&a_2b_1c_3&a_3b_1c_3\end{pmatrix}\\\begin{pmatrix}a_1b_2c_1&a_2b_2c_1&a_3b_2c_1\\a_1b_2c_2&a_2b_2c_2&a_3b_2c_2\\a_1b_2c_3&a_2b_2c_3&a_3b_2c_3\end{pmatrix}\\\begin{pmatrix}a_1b_3c_1&a_2b_3c_1&a_3b_3c_1\\a_1b_3c_2&a_2b_3c_2&a_3b_3c_2\\a_1b_3c_3&a_2b_3c_3&a_3b_3c_3\end{pmatrix}\end{pmatrix}-$$

$$-\begin{pmatrix}\begin{pmatrix}a_1b_1c_1&a_2b_1c_1&a_3b_1c_1\\a_1b_2c_1&a_2b_2c_1&a_3b_2c_1\\a_1b_3c_1&a_2b_3c_1&a_3b_3c_1\end{pmatrix}\\\begin{pmatrix}a_1b_1c_2&a_2b_1c_2&a_3b_1c_2\\a_1b_2c_2&a_2b_2c_2&a_3b_2c_2\\a_1b_3c_2&a_2b_3c_2&a_3b_3c_2\end{pmatrix}\\\begin{pmatrix}a_1b_1c_3&a_2b_1c_3&a_3b_1c_3\\a_1b_2c_3&a_2b_2c_3&a_3b_2c_3\\a_1b_3c_3&a_2b_3c_3&a_3b_3c_3\end{pmatrix}\end{pmatrix}-\begin{pmatrix}\begin{pmatrix}a_1b_1c_1&a_1b_2c_1&a_1b_3c_1\\a_1b_1c_2&a_1b_2c_2&a_1b_3c_2\\a_1b_1c_3&a_1b_2c_3&a_1b_3c_3\end{pmatrix}\\\begin{pmatrix}a_2b_1c_1&a_2b_2c_1&a_2b_3c_1\\a_2b_1c_2&a_2b_2c_2&a_2b_3c_2\\a_2b_1c_3&a_2b_2c_3&a_2b_3c_3\end{pmatrix}\\\begin{pmatrix}a_3b_1c_1&a_3b_2c_1&a_3b_3c_1\\a_3b_1c_2&a_3b_2c_2&a_3b_3c_2\\a_3b_1c_3&a_3b_2c_3&a_3b_3c_3\end{pmatrix}\end{pmatrix}-\begin{pmatrix}\begin{pmatrix}a_1b_1c_1&a_1b_1c_2&a_1b_1c_3\\a_2b_1c_1&a_2b_1c_2&a_2b_1c_3\\a_3b_1c_1&a_3b_1c_2&a_3b_1c_3\end{pmatrix}\\\begin{pmatrix}a_1b_2c_1&a_1b_2c_2&a_1b_2c_3\\a_2b_2c_1&a_2b_2c_2&a_2b_2c_3\\a_3b_2c_1&a_3b_2c_2&a_3b_2c_3\end{pmatrix}\\\begin{pmatrix}a_1b_3c_1&a_1b_3c_2&a_1b_3c_3\\a_2b_3c_1&a_2b_3c_2&a_2b_3c_3\\a_3b_3c_1&a_3b_3c_2&a_3b_3c_3\end{pmatrix}\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}0&0&0\\0&0&d\\0&-d&0\end{pmatrix}\\\begin{pmatrix}0&0&-d\\0&0&0\\d&0&0\end{pmatrix}\\\begin{pmatrix}0&d&0\\-d&0&0\\0&0&0\end{pmatrix}\end{pmatrix}\quad\begin{aligned}d=\ &a_1b_2c_3+a_2b_3c_1+a_3b_1c_2\\&-a_3b_2c_1-a_1b_3c_2-a_2b_1c_3\end{aligned}$$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\wedge\begin{pmatrix}0&b_3&-b_2\\-b_3&0&b_1\\b_2&-b_1&0\end{pmatrix}=2\begin{pmatrix}\begin{pmatrix}0&a_1b_3&-a_1b_2\\-a_1b_3&0&a_1b_1\\a_1b_2&-a_1b_1&0\end{pmatrix}\\\begin{pmatrix}0&a_2b_3&-a_2b_2\\-a_2b_3&0&a_2b_1\\a_2b_2&-a_2b_1&0\end{pmatrix}\\\begin{pmatrix}0&a_3b_3&-a_3b_2\\-a_3b_3&0&a_3b_1\\a_3b_2&-a_3b_1&0\end{pmatrix}\end{pmatrix}+2\begin{pmatrix}\begin{pmatrix}0&-a_1b_3&a_1b_2\\0&-a_2b_3&a_2b_2\\0&-a_3b_3&a_3b_2\end{pmatrix}\\\begin{pmatrix}a_1b_3&0&-a_1b_1\\a_2b_3&0&-a_2b_1\\a_3b_3&0&-a_3b_1\end{pmatrix}\\\begin{pmatrix}-a_1b_2&a_1b_1&0\\-a_2b_2&a_2b_1&0\\-a_3b_2&a_3b_1&0\end{pmatrix}\end{pmatrix}+2\begin{pmatrix}\begin{pmatrix}0&0&0\\a_1b_3&a_2b_3&a_3b_3\\-a_1b_2&-a_2b_2&-a_3b_2\end{pmatrix}\\\begin{pmatrix}-a_1b_3&-a_2b_3&-a_3b_3\\0&0&0\\a_1b_1&a_2b_1&a_3b_1\end{pmatrix}\\\begin{pmatrix}a_1b_2&a_2b_2&a_3b_2\\-a_1b_1&-a_2b_1&-a_3b_1\\0&0&0\end{pmatrix}\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}0&0&0\\0&0&d\\0&-d&0\end{pmatrix}\\\begin{pmatrix}0&0&-d\\0&0&0\\d&0&0\end{pmatrix}\\\begin{pmatrix}0&d&0\\-d&0&0\\0&0&0\end{pmatrix}\end{pmatrix}\quad d=2a_1b_1+2a_2b_2+2a_3b_3$$

The *exterior product* (a.k.a. *wedge product* or *Grassmann product*), symbolized by a wedge ∧, generalizes the cross product and triple product in two important ways: (i) The exterior product can be taken not only of 3-dimensional vectors, but of vectors in any number of dimensions; (ii) The exterior product can be taken not only of vectors, but of (totally antisymmetric) tensors of any rank. The basic idea is to take the tensor product and follow it by antisymmetrization. We will illustrate this idea with three concrete examples and then discuss the product's geometrical significance.

Two 3D Vectors. The exterior product of two 3D vectors a_i and b_j, where $i, j = 1, 2, 3$, is written as $a_i \wedge b_j$ and evaluates to the tensor $a_i b_j - a_j b_i$. The expression $a_i b_j$ is the tensor product (= outer product) of the two vectors, which is then antisymmetrized by subtracting the transposed tensor $a_j b_i$. (Caution: some authors include a factor ½ as part of the antisymmetrization (e.g., [RtR, Ch. 11.6]), but we won't do that here.) As shown in the diagram (top row), the resulting tensor has only three independent components (marked in red) and they correspond to the components of the *cross product*!

What happens if we take this product the other way round? Evaluating $b_i \wedge a_j = b_i a_j - b_j a_i$, we find the *negative* of what we had before. Thus, the exterior product of two vectors *anticommutes*: $a_i \wedge b_j = -\, b_i \wedge a_j$. This is not surprising given that we antisymmetrized the result (and it is also consistent with the fact that the cross product anticommutes: $\vec{a} \times \vec{b} = -\, \vec{b} \times \vec{a}$).

Three 3D Vectors. Next, let's take the exterior product of *three* 3D vectors a_i, b_j, and c_k, where $i, j, k = 1, 2, 3$. Again, we first take the tensor product $a_i b_j c_k$ and then antisymmetrize. But now there are two complications: (i) The product $a_i b_j c_k$ is a rank-3 tensor, a kind of 3×3×3 cubic number scheme, which is hard to write down. In the diagram, we represent it as a column of matrices, where i is the row index of the outer column vector, and j and k are the row and column indices of the inner matrices, respectively. (ii) The antisymmetrization now involves six terms, $a_i \wedge b_j \wedge c_k = a_i b_j c_k + a_j b_k c_i + a_k b_i c_j - a_k b_j c_i -$

$a_i b_k c_j - a_j b_i c_k$, because three indices can be permutated in six different ways: $3! = 6$. We add the three even permutations and subtract the three odd permutations. (Again, we are *not* including a possible factor 1/6 as part of the antisymmetrization.) As shown in the diagram (center row), the resulting tensor has only one independent component (marked in red) and it corresponds to the (scalar) *triple product*!

3D Vector and Antisymmetric 3D Tensor. We can also take the exterior product of a vector a_i and a tensor T_{jk}, provided the latter is antisymmetric ($T_{jk} = -T_{kj}$). The antisymmetry of T_{jk} implies that it has only three independent components. Again, we first take the tensor product $a_i T_{jk}$ and then antisymmetrize: $a_i \wedge T_{jk} = a_i T_{jk} + a_j T_{ki} + a_k T_{ij} - a_k T_{ji} - a_i T_{kj} - a_j T_{ik}$. Making use of the antisymmetry of T_{jk}, this simplifies to $a_i \wedge T_{jk} = 2(a_i T_{jk} + a_j T_{ki} + a_k T_{ij})$. As shown in the diagram (bottom row), the resulting tensor has only one independent component and it corresponds to (twice) the *dot product* of a_i and the three independent components of T_{jk}, which are written as b_n!

What happens if we take this product the other way round? Will it anticommute? No! Evaluating $T_{ij} \wedge a_k = T_{ij} a_k + T_{jk} a_i + T_{ki} a_j - T_{kj} a_i - T_{ik} a_j - T_{ji} a_k$, we find the *same* result as before. So, the exterior product of a vector and an antisymmetric rank-2 tensor *commutes*: $a_i \wedge T_{jk} = T_{ij} \wedge a_k$ (which is also consistent with the fact that the dot product commutes $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$). In general, if the ranks (= number of indices) of both factors are odd, the exterior product anticommutes, but if one or both ranks are even, the product commutes [RtR, Ch. 11.6].

Geometrical Significance. What's so special about the exterior product? The exterior product of two 3D vectors represents the *area* of the parallelogram enclosed by the two vectors along with its orientation in the ambient 3D space. This coincides with the geometrical interpretation of the cross product. The exterior product of three 3D vectors represents the (oriented) *volume* of the parallelepiped enclosed by the three vectors. This coincides with the geometrical interpretation of the triple product. In general, the exterior product of p vectors results in a so-called *p-blade* (or *simple p-vector*), which represents the oriented p-dimensional volume (or area, if $p = 2$) enclosed by these vectors! We refer to these geometrical objects as *area* and *volume elements*.

Why does this geometrical interpretation make sense? The exterior product of two vectors has two major properties: (i) it is linear in each factor, for example, $(\gamma\vec{a}) \wedge \vec{b} = \vec{a} \wedge \left(\gamma\vec{b}\right) = \gamma(\vec{a} \wedge \vec{b})$ and (ii) it anticommutes, $\vec{a} \wedge \vec{b} = -\vec{b} \wedge \vec{a}$, and hence $\vec{a} \wedge \delta\vec{a} = 0$, where γ and δ are scalars. These properties essentially *define* (or axiomatize) what we mean by an area element! If we have a parallelogram defined by the vectors $\vec{a}$ and $\vec{b}$ and (i) we scale up either side $\vec{a}$ or $\vec{b}$ by a factor γ, the enclosed area must go up by the same factor γ and (ii) if both sides are parallel, $\vec{b} = \delta\vec{a}$, the enclosed area must be zero!

Interestingly, we can talk about areas and volumes enclosed by vectors without first defining what the length of a vector is! (To define the length d of a vector a_i, we need a symmetric rank-2 tensor g_{ij}, known as the metric tensor: $d^2 = \sum_{ij} g_{ij} a_i a_j$.)