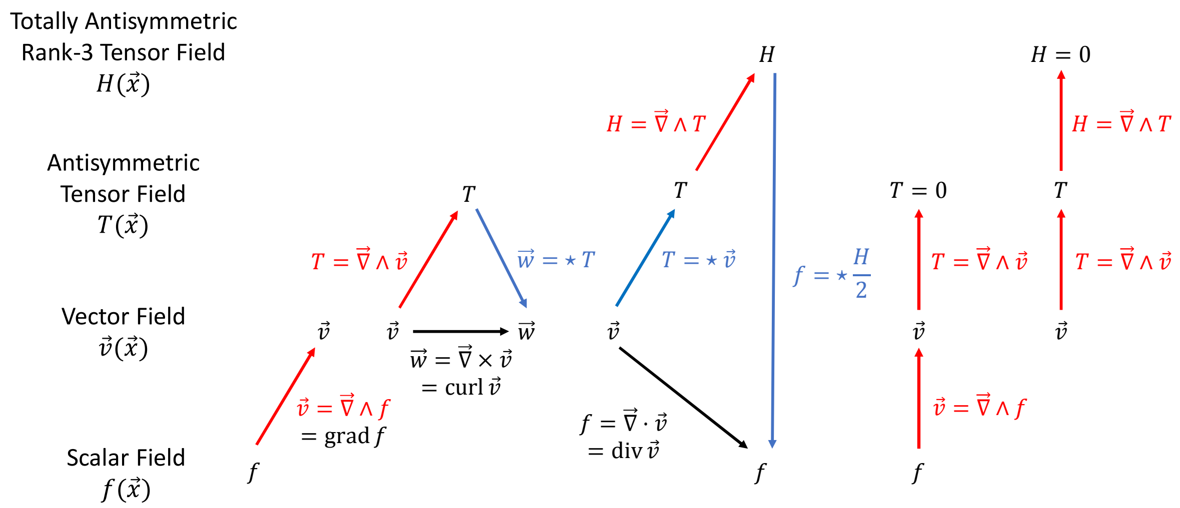
9.8 The Exterior Derivative in 3D Euclidean Space; Gradient, Curl, and Divergence



The *exterior derivative* and the *Hodge dual* are the tools that permit us to unify and generalize the gradient, curl, and divergence operators. In particular, they generalize the curl operator from three to any number of dimensions. We start by discussing these ideas for the familiar 3D Euclidean space rather than a general smooth manifold, thus postponing the abstract concept of a *differential form* until later.

The *exterior product* (a.k.a. *wedge product*) of two vectors is given by the tensor product (= outer product) followed by antisymmetrization, $\vec{u} \wedge \vec{v} \coloneqq \vec{u}\vec{v}^T - \vec{v}\vec{u}^T$, where $\vec{u}$ and $\vec{v}$ are column vectors. See the Appendix "The Exterior Product; Area and Volume Elements" for more details. Spelled out for two 3-dimensional vectors in term of their components, we have

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} (v_1 \quad v_2 \quad v_3) - \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} (u_1 \quad u_2 \quad u_3) = \begin{pmatrix} 0 & u_1 v_2 - v_1 u_2 & u_1 v_3 - v_1 u_3 \\ u_2 v_1 - v_2 u_1 & 0 & u_2 v_3 - v_2 u_3 \\ u_3 v_1 - v_3 u_1 & u_3 v_2 - v_3 u_2 & 0 \end{pmatrix}.$$

The exterior product exists for vectors in any number of dimensions. Moreover, we can take the exterior product of (totally antisymmetric) tensors of any two ranks. Together with the *Hodge dual*, we can re-express (and generalize) the cross product and the dot product. See the Appendix "The Hodge Dual in Euclidean Space" for more details. Using the $\star$ symbol for the Hodge dual, we can write $\vec{u} \times \vec{v} = \star\, (\vec{u} \wedge \vec{v})$ and $\vec{u} \cdot \vec{v} = \frac{1}{2} \star (\star\, \vec{u} \wedge \vec{v}) = \frac{1}{2} \star (\vec{u} \wedge \star\, \vec{v})$, where $\vec{u}$ and $\vec{v}$ are 3-dimensional vectors.

Now, the *exterior derivative* of a vector field is given by $\vec{\nabla} \wedge \vec{v}(\vec{x})$, where the first vector is the differential operator $\vec{\nabla} = (\partial/\partial x_1, \partial/\partial x_2, \cdots, \partial/\partial x_n)^T$. In other words, we take the derivative $\vec{\nabla}\vec{v}^T$ and antisymmetrize the result: $\vec{\nabla}\vec{v}^T - (\vec{\nabla}\vec{v}^T)^T$. Spelled out for a 3-dimensional vector field, we have

$$\begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} (v_1 \quad v_2 \quad v_3) - \left[\begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} (v_1 \quad v_2 \quad v_3) \right]^T = \begin{pmatrix} 0 & \partial v_2/\partial x_1 - \partial v_1/\partial x_2 & \partial v_3/\partial x_1 - \partial v_1/\partial x_3 \\ \partial v_1/\partial x_2 - \partial v_2/\partial x_1 & 0 & \partial v_3/\partial x_2 - \partial v_2/\partial x_3 \\ \partial v_1/\partial x_3 - \partial v_3/\partial x_1 & \partial v_2/\partial x_3 - \partial v_3/\partial x_2 & 0 \end{pmatrix}.$$

The exterior derivative exists for vector fields in any number of dimensions. Moreover, we can take the exterior derivative of a (totally antisymmetric) tensor field of any rank. The exterior derivative of a scalar (rank-0 tensor) field yields a vector field, that of a vector (rank-1 tensor) field yields an antisymmetric tensor field, and that of an antisymmetric (rank-2) tensor field yields a totally antisymmetric rank-3 tensor field (see the red up arrows in the diagram).

The diagram illustrates how the exterior derivative (red arrows) can be related to the gradient, curl, and divergence operators (black arrows) by using the Hodge dual (blue arrows): $\mathrm{grad}\, f = \vec{\nabla} f = \vec{\nabla} \wedge f$, $\mathrm{curl}\, \vec{v} = \vec{\nabla} \times \vec{v} = \star\, (\vec{\nabla} \wedge \vec{v})$, and $\mathrm{div}\, \vec{v} = \vec{\nabla} \cdot \vec{v} = \frac{1}{2} \star (\vec{\nabla} \wedge \star\, \vec{v})$, where $\vec{v}$ is a 3-dimensional vector field.

Taking the exterior derivative twice of any field always yields zero. This is a consequence of the fact that partial derivatives commute [GFKG, Ch. I.4]. In particular, we have $\vec{\nabla} \wedge \vec{\nabla} \wedge f(\vec{x}) = 0$ and $\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{v}(\vec{x}) = 0$, as illustrated on the right-hand side of the diagram. Rewritten in terms of the familiar differential operators, we get the well-known identities $\mathrm{curl}(\mathrm{grad}\, f(\vec{x})) = \vec{0}$ and $\mathrm{div}(\mathrm{curl}\, \vec{v}(\vec{x})) = 0$.

The above calculations can be generalized from Euclidean 3D space to a smooth manifold with any number of dimensions and no need for a metric (= method for measuring distances). This generalization is known as *exterior calculus* and deals with *differential forms,* or *p-forms* for short, which we can think of as totally antisymmetric, totally covariant, rank-p tensor fields on the manifold. For example, a 0-form is a scalar field, a 1-form is a covector field, etc. Note that given the metric-free setup, we are not allowed to identify covectors with vectors and we cannot take the Hodge dual. The exterior derivative of the p-form ω yields a $(p + 1)$-form, which is written as $d\omega$. This appears to be a poor choice of notation because $d\omega$ already means "a little bit of ω", but mathematicians assure us that this clash of notations is intentional [RtR, Ch. 10.3]! In the 3D Euclidean case, taking the exterior derivative of the scalar field $x(\vec{x})$, where $\vec{x} = (x, y, z)^T$, yields $\vec{\nabla} \wedge x = (1,0,0)^T$, that is, "one unit of x". In the generalized case, the same calculation yields dx, that is, "a little bit of x". How can these two results correspond to each other? Well, in the absence of a metric, we cannot measure lengths and thus "one unit" and "a little bit" are the same thing! (Does this make sense?)

Why is this derivative called *exterior*? It turns out that if we integrate the exterior derivative of an $(n - 1)$-form, ω, on an n-dimensional (compact and orientable) manifold, M, with boundary, ∂M, we get the same result as if we integrate the original $(n - 1)$-form over the *exterior* (= boundary) of that manifold. This is the *fundamental theorem of exterior calculus* (a.k.a. *generalized Stokes theorem*): $\int_M d\omega = \int_{\partial M} \omega$ [GFKG, Ch. I.6; RtR, Ch. 12.6]. Special cases of this theorem are (i) the fundamental theorem of calculus: $\int_a^b (df/dx)\, dx = f(b) - f(a)$, (ii) the (old-fashioned) Stokes or curl theorem: $\int_S \mathrm{curl}\, \vec{v}(\vec{x}) \cdot \vec{n} dS = \oint_\Gamma \vec{v}(\vec{x}) \cdot d\vec{\Gamma}$, and (iii) the Gauss or divergence theorem: $\int_V \mathrm{div}\, \vec{v}(\vec{x})\, dV = \oint_S \vec{v}(\vec{x}) \cdot \vec{n} dS$.

What happens if we integrate the exterior derivative of the exterior derivative of an $(n - 2)$-form, $dd\omega = d^2\omega$, over an n-manifold? We get the same result as if we integrate the original $(n - 2)$-form over the boundary of the boundary of that manifold, $\partial\partial M$. But the boundary of the boundary of a smooth manifold is always the empty set, hence $d^2\omega = 0$! This is a key result of exterior calculus that holds for any manifold, regardless of the metric that may or may not be defined on it.