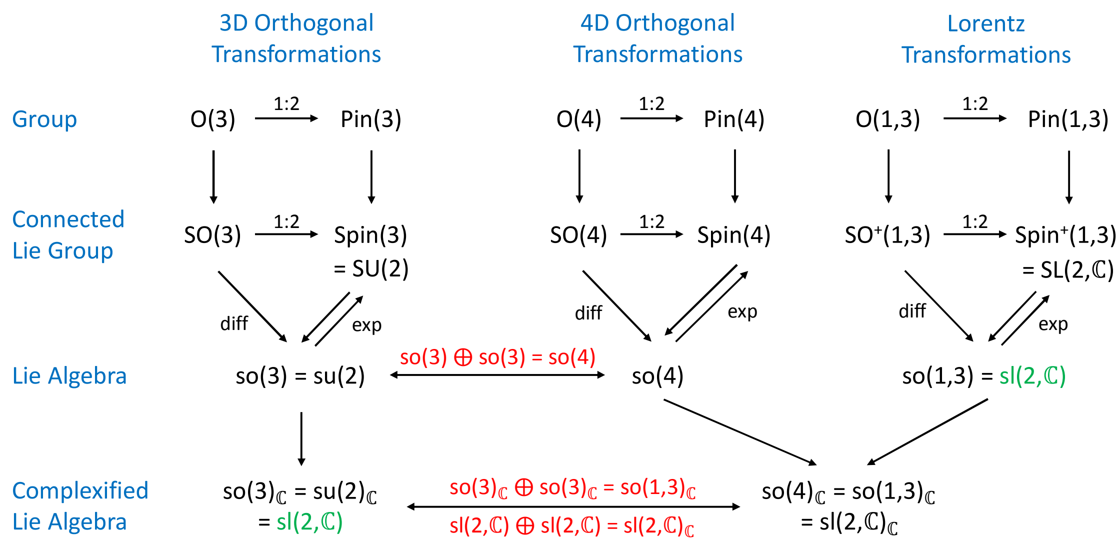
9.9 From Rotation to Lorentz Transformation; Complexification



To find the irreducible representations of the Lorentz group, O(1,3), we take advantage of the relationship between the Lorentz group and the group of rotations in three dimensions, O(3), the irreducible representations of which we already know. As a stepping stone, we make use of the group of rotations in four dimensions, O(4), which acts on 4-component vectors like O(1,3) but is compact like O(3).

To relate the above groups, we need to jump through three hoops:

1. We ignore unconnected components of the groups, retaining only the component that contains the identity element (first to second row in the diagram).
2. We ignore the global structure of the groups, focusing on the Lie algebra instead (second to third row in the diagram). Taking the exponential map of the algebra yields the covering group.
3. Finally, we ignore differences in the metric signature by complexifying (analytically continuing) the Lie algebra (third to fourth row in the diagram). We can recover a desired signature by taking the appropriate real form.

3D Rotation (left-hand side of the diagram). The orthogonal group O(3) has two components, one with the proper (non-reflected) and one with the improper (reflected) rotations. Retaining only the one with proper rotations takes us to SO(3). The associated Lie algebra is so(3). Exponentiating this Lie algebra yields the group SU(2), which double covers SO(3). The associated Lie algebra, su(2), is isomorphic to so(3). To complexify (the defining representation of) so(3), we augment the three real basis generators T_x, T_y, T_z with the three imaginary basis generators $U_x = iT_x$, $U_y = iT_y$, $U_z = iT_z$, making the algebra six (real) dimensional. Whereas the original algebra consisted of the real antisymmetric 3×3 matrices, the complexified algebra consists of the *complex* antisymmetric 3×3 matrices. Similarly, to complexify su(2), we augment the three anti-Hermitian basis generators T_x, T_y, T_z with the three Hermitian basis generators $U_x = iT_x$, $U_y = iT_y$, $U_z = iT_z$. Whereas the original algebra consisted of the traceless anti-Hermitian 2×2 matrices, the complexified algebra consists of all traceless complex 2×2 matrices, hence $\mathrm{su}(2)_\mathbb{C} = \mathrm{sl}(2,\mathbb{C})$. What are the commutation relations of these complexified algebras? We still have $[T_i, T_j] = \varepsilon_{ijk}T_k$ from the original algebra, but then we also have $[U_i, U_j] = [iT_i, iT_j] = -\varepsilon_{ijk}T_k$ and $[T_i, U_j] = [T_i, iT_j] = i\varepsilon_{ijk}T_k = \varepsilon_{ijk}U_k$ (as well as $[U_i, T_j] = \varepsilon_{ijk}U_k$). Amazingly, these are exactly the commutation relations of the Lorentz algebra, so(1,3). In other words, complexifying 3D rotations gives us the special theory of relativity: $\mathrm{so}(3)_\mathbb{C} = \mathrm{so}(1{,}3)$! (We are allowed to complexify the algebras as described above because so(3) ∩ $i$so(3) = su(2) ∩ $i$su(2) = ∅ [QTGR, Ch. 5.5].)

4D Rotation (center of the diagram). The orthogonal group O(4) also has two components. Retaining only the one with the identity element in it takes us to SO(4). The associated Lie algebra is so(4). Exponentiating this Lie algebra yields the group Spin(4), which double covers SO(4). Interestingly, the so(4) algebra breaks up into the direct sum of two so(3) algebras, so(4) = so(3) ⊕ so(3), or, equivalently, so(4) = su(2) ⊕ su(2), revealing a relationship between 4D and 3D rotation. This can be shown by replacing the (plane-rotation) basis generators T_i and U_i of so(4) with the new basis generators $V_i^+ = \frac{1}{2}(T_i + U_i)$ and $V_i^- = \frac{1}{2}(T_i - U_i)$, which have the commutation relations $[V_i^+, V_j^+] = \varepsilon_{ijk}V_k^+$, $[V_i^-, V_j^-] = \varepsilon_{ijk}V_k^-$, and $[V_i^+, V_j^-] = 0$. Similarly, the Spin(4) group breaks up into the direct product of two SU(2) groups, Spin(4) = SU(2) × SU(2). Since the irreducible representations of SU(2) can be labeled with spin values $j = 0$, ½, 1, etc., we conclude that the irreducible representations of Spin(4) can be labeled with *pairs* of spin values (j_1, j_2). The complexified so(4) algebra breaks up into the direct sum of two complexified so(3) algebras: $\mathrm{so}(4)_\mathbb{C} = \mathrm{so}(3)_\mathbb{C} \oplus \mathrm{so}(3)_\mathbb{C}$.

Lorentz Transformation (right-hand side of the diagram). The Lorentz group, O(1,3), has four components, corresponding to the proper/improper rotations and the normal/reversed time directions. Retaining only the component with the identity element in it takes us to the proper orthochronous Lorentz group, SO⁺(1,3). The associated Lie algebra is so(1,3). Exponentiating this Lie algebra yields the group Spin⁺(1,3), which is isomorphic to SL(2,ℂ) and double covers SO⁺(1,3). Whereas the so(1,3) algebra does *not* break up into two subalgebras, its complexification does: $\mathrm{so}(1{,}3)_\mathbb{C} = \mathrm{so}(3)_\mathbb{C} \oplus \mathrm{so}(3)_\mathbb{C}$, or, equivalently, $\mathrm{sl}(2,\mathbb{C})_\mathbb{C} = \mathrm{su}(2)_\mathbb{C} \oplus \mathrm{su}(2)_\mathbb{C}$. This can be shown by replacing the (rotation/boost) basis generators J_i and K_i with the new basis generators $N_i^+ = \frac{1}{2}(J_i + iK_i)$ and $N_i^- = \frac{1}{2}(J_i - iK_i)$, which have the commutation relations $[N_i^+, N_j^+] = i\varepsilon_{ijk}N_k^+$, $[N_i^-, N_j^-] = i\varepsilon_{ijk}N_k^-$, and $[N_i^+, N_j^-] = 0$ [PfS, Ch. 3.7.3]. Note that we combined the old basis generators with *complex* coefficients, an operation that is allowed only after carrying out the complexification. It turns out that the finite-dimensional irreducible representations of Spin⁺(1,3) are in a one-to-one correspondence with those of Spin(4) [Wikipedia: Representation theory of the Lorentz group; the unitarian trick]. Thus, the finite-dimensional irreducible representations of Spin⁺(1,3) can again be labeled with a pair of spin values (j_1, j_2). (However, Spin⁺(1,3) has additional infinite-dimensional representations that Spin(4) doesn't have.)