9.13 Decoding the Proca and Free Maxwell Equations

$\vec{A} \in \mathbb{R}^{1,3}$, Lorentz Rep. $(\frac{1}{2},\frac{1}{2})$	Proca Lagrangian	Proca Equation
	$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\nu$	$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$
Physicist's Notation: signature: $+ - - -$	$\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2} m^2 A^\mu A_\nu$	$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + m^2 A^\nu = 0$
	$\mathcal{L} = -\frac{1}{4}\left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}\right)\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right) + \frac{1}{2} m^2 A^\mu A_\nu$	$\frac{\partial}{\partial x^\mu}\left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}\right) + m^2 A^\nu = 0$
Explicit Metric & Summation: $s_0 = +1, s_1 = s_2 = s_3 = -1$	$\mathcal{L} = -\frac{1}{4}\sum_{i=0}^{3}\sum_{j=0}^{3}\left(s_i \frac{\partial A_j}{\partial x_i} - s_j \frac{\partial A_i}{\partial x_j}\right)\left(s_j \frac{\partial A_j}{\partial x_i} - s_i \frac{\partial A_i}{\partial x_j}\right) + \frac{1}{2} m^2 \sum_{i=0}^{3} s_i A_i^2$	$\sum_{i=0}^{3} \frac{\partial}{\partial x_i}\left(s_i \frac{\partial A_j}{\partial x_i} - s_j \frac{\partial A_i}{\partial x_j}\right) + m^2 A_j = 0$
Fully Expanded:	$\mathcal{L} = \frac{1}{2}\left(\frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial V}{\partial y} + \frac{\partial A_y}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial V}{\partial z} + \frac{\partial A_z}{\partial t}\right)^2$ $-\frac{1}{2}\left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right)^2 - \frac{1}{2}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)^2 - \frac{1}{2}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)^2$ $\frac{1}{2} m^2 (V^2 - A_x^2 - A_y^2 - A_z^2)$	$-\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t}\right) - \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y} + \frac{\partial A_y}{\partial t}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z} + \frac{\partial A_z}{\partial t}\right) + m^2 V = 0$ $\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + m^2 A_x = 0$ $\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial y} + \frac{\partial A_y}{\partial t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right) + m^2 A_y = 0$ $\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial z} + \frac{\partial A_z}{\partial t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right) + m^2 A_z = 0$
EM-Field Vector Notation:	$\mathcal{L} = \frac{1}{2}\lvert\vec{E}\rvert^2 - \frac{1}{2}\lvert\vec{B}\rvert^2 + \frac{1}{2} m^2 (V^2 - \lvert\vec{A}\rvert^2)$	$\vec{\nabla} \cdot \vec{E} + m^2 V = 0$ $-\frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} + m^2 \vec{A} = \vec{0}$

The Proca equation (right-hand side of the diagram) describes a (real) 4-vector field in space time, $\vec{A}(\vec{x})$, that satisfies the symmetries of special relativity, in particular, the field value $\vec{A}$ transforms under the (½, ½) representation of the Lorentz group [PfS, Ch. 6.4]. For $m = 0$, the equation becomes the free Maxwell equation, which describes the electromagnetic potential field (or photon field). The Proca and the free Maxwell equation can be derived from the action principle $\delta \int \mathcal{L}(\vec{A}, \partial\vec{A})\, d^4x = 0$, where $\mathcal{L}$ is the Lagrangian density shown on the left-hand side of the diagram.

Notations. The equations shown at the top of the diagram are written in standard physics notation. The meaning of the upstairs and downstairs indices and the summation convention was already explained for the Klein-Gordon equation. The tensor field $F^{\mu\nu}$ is expanded in the second line, revealing that it is the exterior derivative of the 4-vector field: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The $F^{\mu\nu}$ tensor is composed of the familiar electric and magnetic fields E_i and B_i with $i = x, y, z$:

$$(A^\mu) = \begin{pmatrix} V \\ A_x \\ A_y \\ A_z \end{pmatrix}, \quad (A_\mu) = \begin{pmatrix} V \\ -A_x \\ -A_y \\ -A_z \end{pmatrix}, \quad (F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \quad (F_{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

[QFTGA, Ch. 5.2]. For example, the electric field in the x direction is given by $E_x = F^{10} = \partial^1 A^0 - \partial^0 A^1 = -\partial V/\partial x - \partial A_x/\partial t$ and the magnetic field in the same direction is $B_x = F^{32} = \partial^3 A^2 - \partial^2 A^3 = -\partial A_y/\partial z + \partial A_z/\partial y$.

For the novice it may be clearer to write the equations in a form that makes the sign flips due to the Minkowski metric and the summations explicit. In the second section from the top, we switch to "high-school notation" in which we use only "regular" contravariant vector components and keep all indices downstairs: A_i represents the vector potential components V, A_x, A_y, A_z and x_i represents the space-time coordinates t, x, y, z.

Alternatively, we can fully expand the equations, making all the terms explicit (see the third section from the top). Finally, we can write the equations in terms of electric and magnetic field vectors, $\vec{E} = (E_x, E_y, E_z)^T$ and $\vec{B} = (B_x, B_y, B_z)^T$ (see the bottom of the diagram).

Solutions. The general solution of the Proca equation consists of a superposition of 4-vector-valued plane waves with (angular) frequencies ω and wave vectors $\vec{k} = (k_x, k_y, k_z)^T$:

$$\begin{aligned}\vec{A}(t,x,y,z) = \int \Big(& a_1(\vec{k})\, \vec{\epsilon}_1(\vec{k})\, e^{-i(\omega t - k_x x - k_y y - k_z z)} + a_1^*(\vec{k})\, \vec{\epsilon}_1^*(\vec{k})\, e^{i(\omega t - k_x x - k_y y - k_z z)} \\ & + a_2(\vec{k})\, \vec{\epsilon}_2(\vec{k})\, e^{-i(\omega t - k_x x - k_y y - k_z z)} + a_2^*(\vec{k})\, \vec{\epsilon}_2^*(\vec{k})\, e^{i(\omega t - k_x x - k_y y - k_z z)} \\ & + a_3(\vec{k})\, \vec{\epsilon}_3(\vec{k})\, e^{-i(\omega t - k_x x - k_y y - k_z z)} + a_3^*(\vec{k})\, \vec{\epsilon}_3^*(\vec{k})\, e^{i(\omega t - k_x x - k_y y - k_z z)} \Big)\, d^3k,\end{aligned}$$

where $\omega = \sqrt{m^2 + k_x^2 + k_y^2 + k_z^2}$ and $a_1(\vec{k})$, $a_2(\vec{k})$, $a_3(\vec{k})$ are complex superposition-coefficient functions. The quantities $\vec{\epsilon}_1(\vec{k})$, $\vec{\epsilon}_2(\vec{k})$, and $\vec{\epsilon}_3(\vec{k})$, are wave-vector-dependent polarization vectors. For $\vec{k} = 0$ (rest frame), a common choice is $\vec{\epsilon}_1 = (0,1,0,0)^T$, $\vec{\epsilon}_2 = (0,0,1,0)^T$, $\vec{\epsilon}_3 = (0,0,0,1)^T$. Complex-conjugate plane waves are combined in pairs to form a real-valued wave. After quantization, the three real-valued plane waves correspond to spin-1 particles in three different spin states. The parameter m controls the dispersion relationship $\omega^2 - k_x^2 - k_y^2 - k_z^2 = m^2$ already discussed for the Klein-Gordon equation. In fact, each vector component that solves the Proca equation also solves the Klein-Gordon equation. For the proper normalization of the above solution see [PfS, Ch. 9.4; QFTGA, Ch. 13.2].

Variations. For $m = 0$, the Proca equation becomes the *free Maxwell equation* (a.k.a. the *vacuum Maxwell equation*), that is, the Maxwell equation with zero charge and current densities:

$$\mathcal{L} = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}, \qquad \partial_\mu F^{\mu\nu} = 0.$$

Rewritten in terms of $\vec{E}$ and $\vec{B}$, we obtain the first two (traditional) Maxwell equations. The other two (traditional) Maxwell equations follow from the fact that the exterior derivative of the definition $F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu$ is zero (Bianchi identities) [TM, Vol. 3]. The massless solution has only two transversal polarization vectors: for $\vec{k} = (0,0,k_z)^T$, a common choice is $\vec{\epsilon}_1 = (0,1,0,0)^T$, $\vec{\epsilon}_2 = (0,0,1,0)^T$, $\vec{\epsilon}_3 = 0$. A new symmetry in the massless case asserts that $A'_\mu = A_\mu + \partial_\mu \alpha(\vec{x})$ leaves the Maxwell Lagrangian invariant for any $\alpha(\vec{x})$. This so-called *gauge symmetry* suppresses the longitudinal polarization component [PfS, Ch. 7.1.2; NNQFT, Ch. 5.3.3].

Besides the Proca equation for real fields shown in the diagram, there is also a version for complex fields $A^\mu \in \mathbb{C}$:

$$\mathcal{L} = -\tfrac{1}{2} F^{\mu\nu} F^*_{\mu\nu} + m^2 A^\mu A^*_\mu, \qquad \partial_\mu F^{\mu\nu} + m^2 A^\mu = 0.$$

In the diagram, we used the metric signature $+---$ (as in [QFTGA]), but some authors prefer the signature $-+++$ (as in [TM, Vol. 3]). Then, the equations (for a real field) read

$$\mathcal{L} = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} - \tfrac{1}{2} m^2 A^\mu A_\mu, \qquad \partial_\mu F^{\mu\nu} - m^2 A^\mu = 0.$$