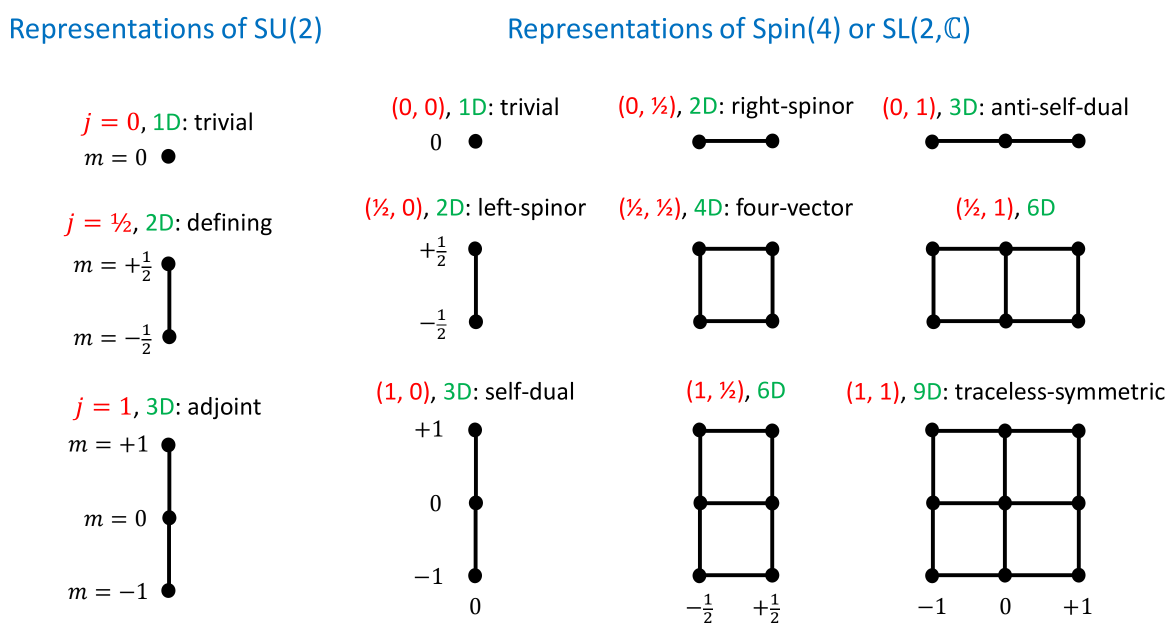
9.10 The Irreducible Representations of the Lorentz Group



What are the irreducible representations of the double cover of the (proper orthochronous) Lorentz group, Spin⁺(1,3) = SL(2,ℂ)? It turns out that the finite-dimensional irreducible representations of SL(2,ℂ) are in a one-to-one correspondence with the irreducible representations of Spin(4), the double cover of the (proper) 4D rotation group [Wikipedia: Representation theory of the Lorentz group; the unitarian trick]. The group Spin(4) has the advantage over SL(2,ℂ) that it decomposes into SU(2) × SU(2), which makes finding the irreducible representations much easier. But before venturing into four dimensions, let's warm up by enumerating the irreducible representations of Spin(3) = SU(2), the double cover of the (proper) 3D rotation group.

Irreducible Representations of SU(2). Because the Lie group SU(2) is simply connected, its representations are in a one-to-one correspondence with those of its algebra, su(2), and it is sufficient to study the representations of the algebra [PfS, Ch. 3.7.3, p. 71]. The corresponding group representations can be obtained by applying the exponential map.

The Lie algebra su(2) is characterized by the commutation relations of its three basis generators T_x, T_y, and T_z, which are $[T_i, T_j] = \varepsilon_{ijk} T_k$ for a common choice of basis. This algebra is of rank one because the maximum number of mutually commuting basis generators is one [GTNut, Ch. VI.2]. By convention, we pick T_z as the generator for the 1-dimensional commuting subalgebra (= Cartan subalgebra). Now, representation theory asserts that the eigenvalues m of $J_z := iT_z$ in a particular representation are in a one-to-one correspondence with the basis vectors of the representation space, that is, we can use m to label the basis vectors [PfS, Ch. 3.5, p. 54]. (The factor i in iT_z is included to make m real.)

The ladder argument shows that the eigenvalues m are spaced apart by 1.0 and range symmetrically from $-j$ to $+j$ (see the Appendix "The Ladder Trick; Raising and Lowering Operators"). So, there is a representation of J_z with a single eigenvalue $m = 0$, another one with two eigenvalues $m = +1/2$ and $-1/2$, yet another one with three eigenvalues $m = +1, 0$ and -1, and so on. These three cases are illustrated on the left-hand side of the diagram. The dots correspond to the eigenvalues m and the vertical lines connecting the dots symbolize the action of the raising and lowering operators. Plots like these are known as *weight diagrams*: the *weights* (= eigenvalues) are represented by dots and the *root vectors* (= raising/lowering operators) correspond to the allowed dot-to-dot movements, in this case just up and down [GTNut, Ch. VI.2]. The dots in the weight diagram not only represent eigenvalues but *also* the basis vectors of the representation space. Thus, the first representation acts on a 1-dimensional representation space, the second one on a 2-dimensional space, the third one on a 3-dimensional space, and so on. We can label each representation by its dimension $d = 1, 2, 3$, etc. or by its largest eigenvalue (weight) $j = 0, ½, 1$, etc. The two schemes are equivalent and related by $d = 2j + 1$.

Some representations have special names. The 1-dimensional representation is known as the *trivial representation*. The 2-dimensional representation is the *defining representation* because SU(2) is defined by its action on 2-dimensional (complex) vectors. The 3-dimensional representation is the *adjoint representation* because the su(2) algebra is three dimensional.

Irreducible Representations of Spin(4) (and SL(2,C)). Because Spin(4) is simply connected, its representations are in a one-to-one correspondence with those of its algebra, so(4).

The Lie algebra so(4) is characterized by the commutation relations of its six basis generators V_x^+, V_y^+, V_z^+, V_x^-, V_y^-, and V_z^-, which are $[V_i^+, V_j^+] = \varepsilon_{ijk} V_k^+$, $[V_i^-, V_j^-] = \varepsilon_{ijk} V_k^-$, and $[V_i^+, V_j^-] = 0$ for a basis in which the decomposition so(4) = su(2) ⊕ su(2) is manifest. This algebra is of rank two because the maximum number of mutually commuting basis generators is two. Let's pick iV_z^+ and iV_z^- as the generators for the 2-dimensional Cartan subalgebra and let's call the eigenvalues of iV_z^+ m_1 and those of iV_z^- m_2. Now, the eigenvalue *pairs* (m_1, m_2) are in a one-to-one correspondence with the basis vectors of the representation space, that is, we can use these pairs to label the basis vectors.

Applying the ladder argument separately to iV_z^+ and iV_z^-, we find that the possible values for m_1 and m_2 are spaced apart by 1.0 and have symmetric ranges: $-j_1 \leq m_1 \leq +j_1$ and $-j_2 \leq m_2 \leq +j_2$. Now, because the weights are pairs, (m_1, m_2), the weight diagram becomes 2-dimensional. In the above diagram, m_1 is drawn on the vertical and m_2 on the horizontal axis. The raising and lowering operators acting on iV_z^+ move us to the dot above and below, whereas the raising and lowering operators acting on iV_z^- move us right and left, effectively turning the ladder into a "jungle gym". Instead of two there are now four root vectors, one pointing up, one down, one left and one right. Some of the possible weight diagrams are illustrated on the right-hand side. The total number of dots in the diagram equals the dimension of the representation. Note that the dimension does not provide a unique label in this case, hence we use the largest weight, (j_1, j_2), as the label.

Some representations have special names. The 1-dimensional (0, 0) representation is known as the *trivial representation*. The 2-dimensional representations (½, 0) and (0, ½) are the *left-chiral* and *right-chiral Weyl spinor representations*, respectively. The 3-dimensional representations (1, 0) and (0, 1) are the *self-dual* and *anti-self-dual representations*, respectively. The 4-dimensional (½, ½) representation is the *4-vector representation*. The 9-dimensional (1, 1) representation is the *traceless-symmetric tensor representation*. Incidentally, the *defining representation* of SL(2,ℂ) is (½, 0) and the *adjoint representation* is (1, 0) ⊕ (0, 1).