9.11 Decoding the Klein-Gordon Equation

$\phi \in \mathbb{R}$, Lorentz Rep. $(0,0)$	Klein-Gordon Lagrangian	Klein-Gordon Equation
Physicist's Notation: signature: $+---$	$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$ $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$ $\mathcal{L} = \frac{1}{2}\frac{\partial \phi}{\partial x^\mu}\frac{\partial \phi}{\partial x_\mu} - \frac{1}{2} m^2 \phi^2$	$(\Box + m^2)\phi = 0$ $(\partial_\mu \partial^\mu + m^2)\phi = 0$ $\frac{\partial}{\partial x^\mu}\frac{\partial \phi}{\partial x_\mu} + m^2 \phi = 0$
Explicit Metric & Summation: $s_0 = +1, s_1 = s_2 = s_3 = -1$	$\mathcal{L} = \sum_{i=0}^{3} s_i \frac{1}{2}\left(\frac{\partial \phi}{\partial x_i}\right)^2 - \frac{1}{2} m^2 \phi^2$	$\sum_{i=0}^{3} s_i \frac{\partial^2 \phi}{\partial x_i^2} + m^2 \phi = 0$
Vector-Matrix Notation: $\eta = \mathrm{diag}(+1,-1,-1,-1)$	$\mathcal{L} = \frac{1}{2}\left(\vec{\nabla}\phi\right)^T \eta \, \vec{\nabla}\phi - \frac{1}{2} m^2 \phi^2$	$\left(\vec{\nabla}\right)^T \eta \, \vec{\nabla}\phi + m^2 \phi = 0$
Fully Expanded:	$\mathcal{L} = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial y}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial z}\right)^2 - \frac{1}{2} m^2 \phi^2$	$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} + m^2 \phi = 0$

The Klein-Gordon equation (right-hand side of the diagram) describes a (real) scalar field in space time, $\phi(\vec{x})$, that satisfies the symmetries of special relativity, in particular, the field value ϕ transforms under the (0, 0), or trivial, representation of the Lorentz group [PfS, Ch. 6.2]. The Klein-Gordon equation can be derived from the action principle $\delta \int \mathcal{L}(\phi, \partial\phi)\, d^4x = 0$, where $\mathcal{L}$ is the Lagrangian density shown on the left-hand side of the diagram [TM, Vol. 3]. The Higgs field is an example of a scalar field.

Notations. The equations at the top of the diagram are written in standard physics notation. The $\Box$ symbol, known as the d'Alembertian, is expanded in the second line revealing that it is the second derivative $\partial_\mu \partial^\mu$, a kind of Laplacian with Lorentzian signature. Space-time coordinates with upstairs (contravariant) indices, x^μ, represent the usual time and space coordinates, whereas coordinates with downstairs (covariant) indices, x_μ, represent the time and space coordinates with the signs of the latter flipped (i.e., x^μ multiplied by the "hidden" metric tensor $\mathrm{diag}(1,-1,-1,-1)$). Derivatives with respect to time and space coordinates are written as $\partial_\mu = \partial/\partial x^\mu$ and $\partial^\mu = \partial/\partial x_\mu$ (note the index positions). Moreover, standard physics notation implies a summation over repeated indices (Einstein summation convention). Spelled out explicitly, the space-time vectors and their derivatives are:

$$(x^\mu) = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \quad (x_\mu) = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ -x \\ -y \\ -z \end{pmatrix}, \quad (\partial_\mu) = \begin{pmatrix} \partial/\partial x^0 \\ \partial/\partial x^1 \\ \partial/\partial x^2 \\ \partial/\partial x^3 \end{pmatrix} = \begin{pmatrix} \partial/\partial t \\ \partial/\partial x \\ \partial/\partial y \\ \partial/\partial x \end{pmatrix}, \quad (\partial^\mu) = \begin{pmatrix} \partial/\partial x_0 \\ \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} = \begin{pmatrix} \partial/\partial t \\ -\partial/\partial x \\ -\partial/\partial y \\ -\partial/\partial x \end{pmatrix}.$$

For the novice it may be clearer to write the equations in a form that makes the sign flips due to the Minkowski metric and the summation explicit. In the second section from the top, we switch to "high-school notation" in which we use only "regular" contravariant vector components and keep all indices downstairs (and in Latin). For example, x_i represents the space-time coordinates t, x, y, z.

The same equations can also be written in vector-matrix notation, as shown in the third section from the top. For this purpose, we define the 4×4 matrix η (the Minkowski metric tensor) and the 4-component column vector $\overset{\rightharpoonup}{\nabla} = (\partial/\partial t, \vec{\nabla})^T$ (the space-time derivative):

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \overset{\rightharpoonup}{\nabla} = \begin{pmatrix} \partial/\partial t \\ \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}.$$

Note that we mark 3-dimensional vectors with an *arrow* and 4-dimensional space-time vectors with a *harpoon* to distinguish the two types of vectors.

Finally, we can fully expand the equations, making all the terms explicit, as shown at the bottom of the diagram.

Solutions. The general solution of the Klein-Gordon equation consists of a superposition of plane waves with (angular) frequencies ω and wave vectors $\vec{k} = (k_x, k_y, k_z)^T$:

$$\phi(t,x,y,z) = \int \left(a(\vec{k})\, e^{-i(\omega t - k_x x - k_y y - k_z z)} + a^*(\vec{k})\, e^{i(\omega t - k_x x - k_y y - k_z z)} \right) d^3k,$$

where $\omega = \sqrt{m^2 + k_x^2 + k_y^2 + k_z^2}$ and $a(\vec{k})$ is a complex superposition-coefficient function. Complex-conjugate plane waves are combined in pairs to form a real-valued wave. For the proper normalization of the above solution see [PfS, Ch. 9.6; QFTGA, Ch. 11.2].

The parameter m controls the dispersion relationship: $\omega^2 - k_x^2 - k_y^2 - k_z^2 = m^2$. For $m = 0$, the wave does not disperse and always moves at the speed of light ($c = 1$). For $m \neq 0$, the wave has the ability to stand still while oscillating in time. After quantizing the field, the parameter m represents the mass of the quanta. The frequency ω and the wave vector $\vec{k}$ represent the energy and momentum of the quanta, respectively ($\hbar = 1$). For $m = 0$, the quanta always move at the speed of light. For $m \neq 0$, the quanta have the ability to be at rest while possessing a nonzero energy ($E = \omega = m$).

Variations. For $m = 0$, the Klein-Gordon equation becomes the wave equation:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi, \qquad \partial_\mu \partial^\mu \phi = 0.$$

Besides the Klein-Gordon equation for real fields shown in the diagram, there is also a version for complex fields $\phi \in \mathbb{C}$:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi, \qquad \partial_\mu \partial^\mu \phi + m^2 \phi = 0.$$

The complex version of the Klein-Gordon equation can be decomposed into two copies of the real version [QFTGA, Ch. 7.6; PfS, Ch. 6.2.1]. The general solution is like the one shown above, but with $a^*(\vec{k})$ replaced by an independent complex superposition-coefficient function $b(\vec{k})$.

In the diagram, we used the metric signature $+---$ (as in [QFTGA]), but some authors prefer the signature $-+++$ (as in [TM, Vol. 3]). Then, the equations (for a real field) read

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi + \tfrac{1}{2} m^2 \phi^2, \qquad \partial_\mu \partial^\mu \phi - m^2 \phi = 0.$$