9.12 Decoding the Dirac Equation

$\psi \in \mathbb{C}^4$, Lorentz Rep. $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	Dirac Lagrangian	Dirac Equation
Physicist's Notation:	$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$	$(i\partial\!\!\!/ - m)\psi = 0$
	$\mathcal{L} = \psi^\dagger\gamma^0(i\gamma^\mu\partial_\mu - m)\psi$	$(i\gamma^\mu\partial_\mu - m)\psi = 0$
	$\mathcal{L} = i\psi^\dagger\gamma^0\gamma^\mu\frac{\partial\psi}{\partial x^\mu} - m\psi^\dagger\gamma^0\psi$	$i\gamma^\mu\frac{\partial\psi}{\partial x^\mu} - m\psi = 0$
Explicit Summation:	$\mathcal{L} = i\sum_{j=0}^{3}\sum_{a=1}^{4}\sum_{b=1}^{4}\sum_{c=1}^{4}\psi_a^*\gamma_{0ab}\gamma_{jbc}\frac{\partial\psi_c}{\partial x_j} - m\sum_{a=1}^{4}\sum_{b=1}^{4}\psi_a^*\gamma_{0ab}\psi_b$	$i\sum_{j=0}^{3}\sum_{b=1}^{4}\gamma_{jab}\frac{\partial\psi_b}{\partial x_j} - m\psi_a = 0$
Vector-Matrix Notation:	$\mathcal{L} = i\psi^\dagger\frac{\partial\psi}{\partial t} + i\psi^\dagger\gamma_0\gamma_1\frac{\partial\psi}{\partial x} + i\psi^\dagger\gamma_0\gamma_2\frac{\partial\psi}{\partial y} + i\psi^\dagger\gamma_0\gamma_3\frac{\partial\psi}{\partial z} - m\psi^\dagger\gamma_0\psi$	$i\gamma_0\frac{\partial\psi}{\partial t} + i\gamma_1\frac{\partial\psi}{\partial x} + i\gamma_2\frac{\partial\psi}{\partial y} + i\gamma_3\frac{\partial\psi}{\partial z} - m\psi = 0$
Fully Expanded: chiral representation: $\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$	$\begin{aligned}\mathcal{L} = {}& i\psi_3^*\frac{\partial\psi_3}{\partial t} + i\psi_3^*\frac{\partial\psi_4}{\partial x} + \psi_3^*\frac{\partial\psi_4}{\partial y} + i\psi_3^*\frac{\partial\psi_3}{\partial z} \\ &+ i\psi_4^*\frac{\partial\psi_4}{\partial t} + i\psi_4^*\frac{\partial\psi_3}{\partial x} - \psi_4^*\frac{\partial\psi_3}{\partial y} - i\psi_4^*\frac{\partial\psi_4}{\partial z} \\ &+ i\psi_1^*\frac{\partial\psi_1}{\partial t} - i\psi_1^*\frac{\partial\psi_2}{\partial x} - \psi_1^*\frac{\partial\psi_2}{\partial y} - i\psi_1^*\frac{\partial\psi_1}{\partial z} \\ &+ i\psi_2^*\frac{\partial\psi_2}{\partial t} - i\psi_2^*\frac{\partial\psi_1}{\partial x} + \psi_2^*\frac{\partial\psi_1}{\partial y} + i\psi_2^*\frac{\partial\psi_2}{\partial z} \\ &- m(\psi_3^*\psi_1 + \psi_4^*\psi_2 + \psi_1^*\psi_3 + \psi_2^*\psi_4)\end{aligned}$	$\begin{aligned} & i\frac{\partial\psi_3}{\partial t} + i\frac{\partial\psi_4}{\partial x} + \frac{\partial\psi_4}{\partial y} + i\frac{\partial\psi_3}{\partial z} - m\psi_1 = 0 \\ & i\frac{\partial\psi_4}{\partial t} + i\frac{\partial\psi_3}{\partial x} - \frac{\partial\psi_3}{\partial y} - i\frac{\partial\psi_4}{\partial z} - m\psi_2 = 0 \\ & i\frac{\partial\psi_1}{\partial t} - i\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial y} - i\frac{\partial\psi_1}{\partial z} - m\psi_3 = 0 \\ & i\frac{\partial\psi_2}{\partial t} - i\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_1}{\partial y} + i\frac{\partial\psi_2}{\partial z} - m\psi_4 = 0\end{aligned}$

The Dirac equation (right-hand side of the diagram) describes a (complex) 4-component spinor field in space time, $\psi(\vec{x})$, that satisfies the symmetries of special relativity, in particular, the field value ψ transforms under the (½, 0) ⊕ (0, ½) representation of the (double cover of the) Lorentz group [PfS, Ch. 6.3]. The Dirac equation can be derived from the action principle $\delta\int\mathcal{L}(\psi,\partial\psi)\,d^4x = 0$, where $\mathcal{L}$ is the Lagrangian density shown on the left-hand side of the diagram. The electron/positron field is an example of such a spinor field.

Notations. The equations shown at the top of the diagram are written in standard physics notation. The $\partial\!\!\!/$ symbol with the *Feynman slash* is expanded in the second line revealing "hidden" gamma matrices: $\partial\!\!\!/ = \gamma^\mu\partial_\mu$ (the slash should be diagonal, but I don't know how to do that in MS Word). The *conjugate spinor* with the overbar expands to $\bar{\psi} = \psi^\dagger\gamma^0$. The meaning of the upstairs and downstairs indices and the summation convention was already explained for the Klein-Gordon equation. Note that the hybrid notation γ^μ symbolizes four objects ($\mu = 0, 1, 2, 3$) each of which is a 4×4 matrix. A common choice for the gamma matrices, known as the *Dirac* or *mass representation*, is [PfS, Ch. 8.10]:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},\ \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},\ \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Another choice, known as the *Weyl* or *chiral representation*, is [QFTGA, Ch. 36.2]:

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},\ \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},\ \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(Caution: some authors use the opposite sign for γ^1, γ^2, and γ^3 [e.g., GTNut, Ch. VII.5].)

The gamma matrices must be elements of a Clifford algebra with Lorentzian signature, that is, they must satisfy $(\gamma^0)^2 = 1$, $(\gamma^1)^2 = -1$, $(\gamma^2)^2 = -1$, $(\gamma^3)^2 = -1$, and $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$. Why? Under these conditions, the square $(\gamma^0\partial_0 + \gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3)^2$ evaluates to the simple expression $\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$, where all the mixed products canceled! More compactly, we can write this as $(\gamma^\mu\partial_\mu)^2 = \partial^\mu\partial_\mu$ or if we want to be slick as $\not{\partial}^2 = \Box$. With this relationship, we can rewrite the $(\partial^\mu\partial_\mu + m^2)$ operator in the Klein-Gordon equation as $(\gamma^\mu\partial_\mu + im)(\gamma^\mu\partial_\mu - im)$, revealing that the Dirac equation is, in some sense, the "square root" of the Klein-Gordon equation!

For the novice it may be clearer to write the equations in "high-school notation" in which we use only "regular" contravariant vector components, keep all indices downstairs, and spell out all summations explicitly (see the second section from the top). Note that the gamma object now has three indices: one space-time index (ranging from 0 to 3) and two spinor indices (ranging from 1 to 4).

Alternatively, we can write the equations in vector-matrix notation (see the third section from the top).

Finally, we can fully expand the equations, making all the terms explicit, as shown at the bottom of the diagram (where the chiral representation of the gamma matrices was used).

Solutions. The general solution of the Dirac equation consists of a superposition of spinor-valued plane waves with (angular) frequencies ω and wave vectors $\vec{k} = (k_x, k_y, k_z)^T$:

$$\psi(t,x,y,z) = \int \big(c_u(\vec{k})\, u_u(\vec{k})\, e^{-i(\omega t - k_x x - k_y y - k_z z)} + d_u(\vec{k})\, v_u(\vec{k})\, e^{i(\omega t - k_x x - k_y y - k_z z)} \\ + c_d(\vec{k})\, u_d(\vec{k})\, e^{-i(\omega t - k_x x - k_y y - k_z z)} + d_d(\vec{k})\, v_d(\vec{k})\, e^{i(\omega t - k_x x - k_y y - k_z z)}\big)\, d^3k,$$

where $\omega = \sqrt{m^2 + k_x^2 + k_y^2 + k_z^2}$. The quantities $u_u(\vec{k})$, $v_u(\vec{k})$, $u_d(\vec{k})$, and $v_d(\vec{k})$, are wave-vector-dependent spinors. For $\vec{k} = 0$ (rest frame) and the chiral representation, a common choice is $u_u = (1,0,1,0)^T$, $v_u = (-1,0,1,0)^T$, $u_d = (0,1,0,1)^T$, $v_d = (0,-1,0,1)^T$ [PfS, p. 200]. After quantization, the four terms in the above solution correspond to particles with spin up (+½), antiparticles with spin up, particles with spin down (−½), and antiparticles with spin down, respectively. Note that all solutions have a left-chiral part, $(\psi_1, \psi_2) \neq 0$, *and* a right-chiral part, $(\psi_3, \psi_4) \neq 0$. The complex superposition-coefficient functions $c_u(\vec{k})$, $d_u(\vec{k})$, $c_d(\vec{k})$, and $d_d(\vec{k})$ can be chosen arbitrarily (the result does not have to be real valued). The parameter m controls the dispersion relationship $\omega^2 - k_x^2 - k_y^2 - k_z^2 = m^2$ already discussed for the Klein-Gordon equation. In fact, each spinor component that solves the Dirac equation also solves the Klein-Gordon equation. For the proper normalization of the above solution see [PfS, Ch. 8.10.1; QFTGA, Ch. 36.4].

Variations. For $m = 0$, the Dirac equation breaks up into two independent sets of equations, known as the *Weyl equations*. For the chiral representation, we have

$$\mathcal{L} = i\psi_R^\dagger(\partial_0 + \sigma^i\partial_i)\psi_R + i\psi_L^\dagger(\partial_0 - \sigma^i\partial_i)\psi_L, \qquad (\partial_0 + \sigma^i\partial_i)\psi_R = 0, (\partial_0 - \sigma^i\partial_i)\psi_L = 0,$$

where σ^i are the Pauli matrices and $\psi_L = (\psi_1, \psi_2)^T$, $\psi_R = (\psi_3, \psi_4)^T$ are the Weyl spinors making up the Dirac spinor $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$. After quantization, the ψ_L field corresponds to left-chiral particles and the ψ_R field to right-chiral particles. Note that in the massless case, solutions with a single chirality are possible.