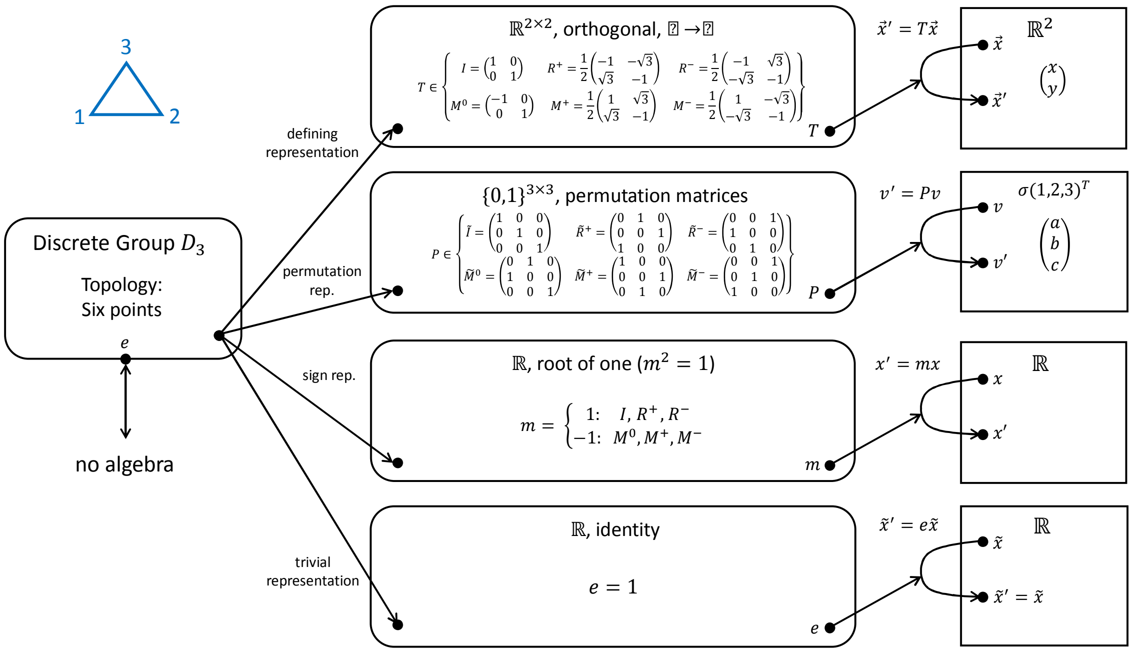
4. Reflection and Parity

4.1 D3: The Symmetry Group of the Equilateral Triangle; Semidirect Product



So far, we have discussed Lie groups, that is, continuous groups with infinitely many elements. Now, we'll turn to a *discrete group* with a *finite* number of elements, namely, the group of symmetries of the equilateral triangle known as the *dihedral group of order three*, D3. This group is rich enough to illustrate most features of discrete groups, yet simple enough to be suitable for beginners. In fact, this is the first group I was introduced to in high school! Moreover, the group $\mathbb{Z}_2$, which is important for understanding the relationship between O(n) and SO(n), is contained in D3 as a subgroup.

Imagine an equilateral triangle with vertices 1, 2, and 3 located at the (x, y) coordinates $(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$, $(+\frac{1}{2}, -\frac{1}{2\sqrt{3}})$, and $(0, \frac{1}{\sqrt{3}})$, respectively (see the diagram). What transformations can we perform on this triangle without changing its appearance? We can rotate it counterclockwise by 120° ($= R^+$) and we can rotate it clockwise by 120° ($= R^-$). These are all possible rotations: rotating counterclockwise by 2×120° is the same thing as rotating clockwise by 120° ($R^+R^+ = R^-$); rotating counterclockwise by 3×120° is the same thing as doing nothing ($R^+R^+R^+ = I$). Besides rotating, we can reflect (or mirror) the triangle about the y axis ($= M^0$), about the 120°-rotated y axis ($= M^+$), and about the −120°-rotated y axis ($= M^-$). In total, there are six transformations, which constitute the six elements of the group D3. The defining representation of this group is given by the real 2×2 matrices $T \in \{I, R^+, R^-, M^0, M^+, M^-\}$ that transform the (x, y) coordinates of the triangle (see the top branch of the diagram).

Interestingly, there is another way of looking at these triangle transformations, which leads to a conceptually different, but mathematically isomorphic, representation. Instead of transforming (x, y) coordinates, we permute vertices! The triangle has three vertices and thus there are 3×2×1=6 possible permutations. For example, the counterclockwise rotation can be represented by the permutation $\tilde{R}^+$: $(1,2,3) \to (3,1,2)$, the y-axis reflection can be represented by $\widetilde{M}^0$: $(1,2,3) \to (2,1,3)$, and so on. This representation is given by the 3×3 permutation matrices $P \in \{\tilde{I}, \tilde{R}^+, \tilde{R}^-, \widetilde{M}^0, \widetilde{M}^+, \widetilde{M}^-\}$, which act on the "vertex-number vector" v (second branch of the diagram). The group of permutations of n objects is known as the *symmetric group of order n*, S_n. Therefore, we have the isomorphism $D_3 = S_3$.

Discrete groups don't have Lie algebras associated with them because there are no infinitesimally small transformations. Does this mean that they also don't have generators? They do, but the meaning of the term "generator" is slightly different from the Lie case. The group D_3 has two generators, which are given, for example, by the elements R^+ and M^0. These elements generate all six transformations when applied repeatedly. For example: $R^- = R^+R^+$, $M^+ = M^0R^+$, $M^- = R^+M^0$, and $I = R^+R^+R^+ = M^0M^0$ (see the multiplication table below). Instead of R^+ and M^0, we could have picked any rotation and reflection element as the generators. Note that the generators of a discrete group correspond to the small *transformations* $I + \varepsilon X$ in the Lie group, not the generator X in the Lie algebra.

$\boldsymbol{I}$	$\boldsymbol{R^+}$	$\boldsymbol{R^-}$	$\boldsymbol{M^0}$	$\boldsymbol{M^+}$	$\boldsymbol{M^-}$
$\boldsymbol{R^+}$	R^-	I	M^-	M^0	M^+
$\boldsymbol{R^-}$	I	R^+	M^+	M^-	M^0
$\boldsymbol{M^0}$	M^+	M^-	I	R^+	R^-
$\boldsymbol{M^+}$	M^-	M^0	R^-	I	R^+
$\boldsymbol{M^-}$	M^0	M^+	R^+	R^-	I

The group D_3 has a subgroup of rotations consisting of the three elements $\{I, R^+, R^-\}$ and a subgroup of reflections consisting, for example, of the two elements $\{I, M^0\}$. These subgroups are denoted $\mathbb{Z}_3$ and $\mathbb{Z}_2$, respectively. We can recover all the elements of D_3 by taking the Cartesian product of the two subgroups: $\{I, R^+, R^-\} \times \{I, M^0\} = \{(I,I), (R^+,I), (R^-,I), (I,M^0), (R^+,M^0), (R^-,M^0)\}$. If we interpret the pairs of the Cartesian product (a, b) as $a \circ b$ (apply a after b), we can identify: $(I,I) \to I$, $(R^+,I) \to R^+$, $(R^-,I) \to R^-$, $(I,M^0) \to M^0$, $(R^+,M^0) \to M^-$, and $(R^-,M^0) \to M^+$.

How are we to compose two elements of the Cartesian product? We could try to do it componentwise: $(a,b) \circ (\alpha,\beta) = (a \circ \alpha, b \circ \beta)$. For example, we could compose $(R^+,I)(R^-,I)$ as $(R^+R^-, II) = (I,I)$, which gives the correct result, because $R^+R^- = I$. This type of product is known as the *direct product*. Let's try another example: $(R^+,M^0)(R^-,M^0) = (R^+R^-, M^0M^0) = (I,I)$, but this is certainly not correct because $M^-M^+ = R^- \neq I$! What went wrong? To correctly compose the pairs (a,b) and (α,β), we have to calculate $(a \circ b) \circ (\alpha \circ \beta)$, but if we use the direct product, we calculate $(a \circ \alpha) \circ (b \circ \beta)$, which is not necessarily the same! To try and fix this, let's insert the identity $b^{-1}b$ as follows: $a \circ b \circ \alpha \circ \beta = a \circ b \circ \alpha \circ (b^{-1} \circ b) \circ \beta = (a \circ [b \circ \alpha \circ b^{-1}]) \circ (b \circ \beta)$. Now, this looks almost like the direct product, except that we have $b\alpha b^{-1}$ instead of α! In conclusion, we have to compose the pairs as follows: $(a,b) \circ (\alpha,\beta) = (a \circ b\alpha b^{-1}, b \circ \beta)$. This type of product is known as the *semidirect product*. Let's try our second example again: $(R^+,M^0)(R^-,M^0) = (R^+[M^0R^-M^0], M^0M^0) = (R^+R^+, I) = (R^-, I)$, which is the correct result! The semidirect product of $\mathbb{Z}_3$ and $\mathbb{Z}_2$ is written as $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$. Thus, we have the isomorphism $D_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$.

The group D_3 has three irreducible representations: the defining representation, which we already discussed, the *sign representation* (third branch of the diagram) and, as always, the *trivial representation* (bottom branch of the diagram). For an explanation of why there are three irreducible representations, see [GTNut, Ch. II.3] and the Quanta magazine article https://www.quantamagazine.org/the-useless-perspective-that-transformed-mathematics-20200609/.