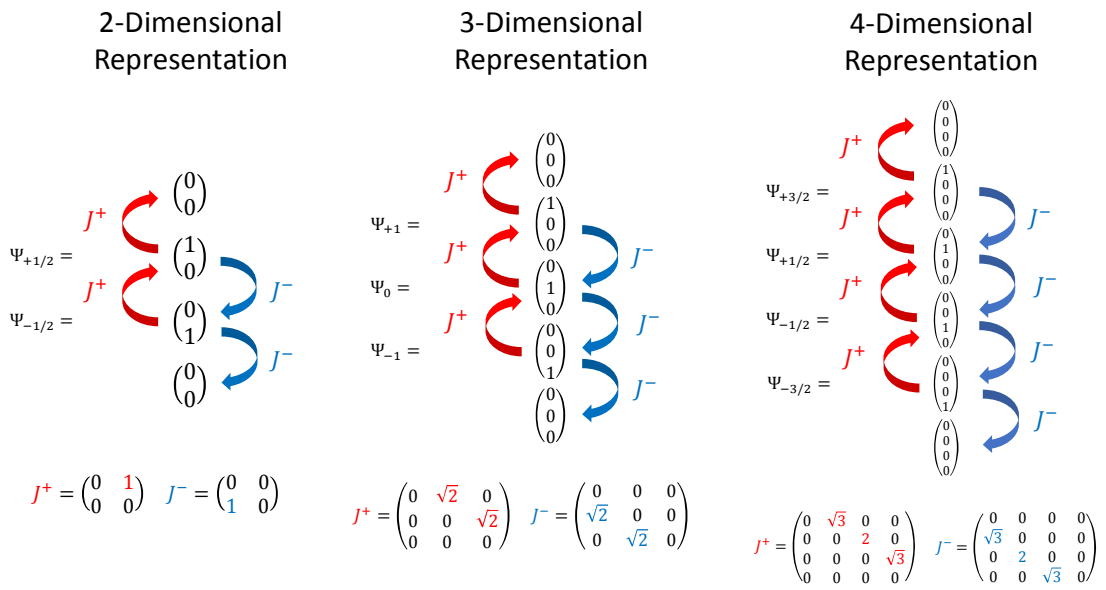
5.2 The Ladder Trick; Raising and Lowering Operators



Let's assume that we are given a Lie algebra and the only thing we know about it is its commutation relations. What can we infer from that? For so(2) and u(1) the commutation relations are trivial (there is only one basis generator, which commutes with itself) and there is nothing interesting we can infer from that. For so(3), su(2), and sp(1), however, there are three nontrivial commutation relations: $[J_x,J_y] = iJ_z$, $[J_y,J_z] = iJ_x$, and $[J_z,J_x] = iJ_y$. Interestingly, from these relations alone, we can conclude that the eigenvalues of J_z, or any other basis generator, are spaced apart by exactly 1.0! In other words, the eigenvalues form a ladder with a rung spacing of 1.0.

How can we demonstrate this? We construct raising and lowering operators that when applied to a given eigenvector produce the eigenvector with the next higher and the eigenvector with the next lower eigenvalue, respectively. Let's focus on the eigenvectors and eigenvalues of J_z (we can do the same trick with any of the basis generators). The two commutation relations with J_z in the commutator look very similar: $[J_z,J_x] = iJ_y$ and $[J_z,J_y] = -iJ_x$. We combine them in two different ways: $[J_z,J_x + iJ_y] = J_x + iJ_y$ and $[J_z,J_x - iJ_y] = -J_x + iJ_y$. After introducing the new names $J^+ = J_x + iJ_y$ and $J^- = J_x - iJ_y$, we can restate the two combined commutation relations more compactly: $[J_z,J^+] = J^+$ and $[J_z,J^-] = -J^-$. It turns out that J^+ and J^- are exactly the raising and lowering operators we are looking for. Why? I don't have a good intuitive explanation for this fact, but, as we will see, it is straightforward to demonstrate this mathematically. (Note that J^+ and J^- are *not* elements of the Lie algebra, which permits only *real* combinations of basis generators. Instead, they are elements of the *complexified Lie algebra*, which is different from so(3) = su(2) = sp(1).)

Let's show that J^+ raises the eigenvalue by 1.0 (analogously, one can show that J^- lowers the eigenvalue by 1.0). We start by assuming that we already have an eigenvector Ψ_m with eigenvalue m, that is, the eigenequation $J_z\Psi_m = m\Psi_m$ is satisfied. Then, we apply the raising operator to both sides of this equation $J^+J_z\Psi_m = J^+m\Psi_m$. Next, we swap the operators J^+J_z by using our combined commutation

relation $[J_z, J^+] = J_z J^+ - J^+ J_z = J^+$. So, we get $(J_z J^+ - J^+)\Psi_m = J^+ m \Psi_m$. Rearranging the terms yields $J_z J^+ \Psi_m = (m+1) J^+ \Psi_m$. What is this? It is a new eigenequation that says that $J^+\Psi_m$ is an eigenvector of J_z with eigenvalue $m+1$. Not only is this the next higher eigenvalue, but its value went up by exactly 1.0, just what we set out to show!

But that's not all. We know that an n-dimensional representation consists of $n \times n$ matrices and such matrices have n eigenvalues. Moreover, because the map $J_3 \to -J_3,\ J^+ \to J^-,\ J^- \to J^+$ is a symmetry of the commutation relations, the absolute value of the largest and smallest eigenvalue must be the same. Combining these facts with the spacing of 1.0, we can infer all the eigenvalues of all the representations: for the 2-dimensional representation they must be −½ and +½, for the 3-dimensional representation they must be −1, 0 +1, and so on.

What is the explicit matrix form of the raising and lowering operator? Let's try this out for the 3-dimensional representation: $J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $J_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$, and $J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Combining J_x and J_y results in $J^+ = J_x + iJ_y = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $J^- = J_x - iJ_y = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. These are the raising and lowering operators for J_z.

We already know the eigenvectors and eigenvalues of J_z. For example, the eigenvector with the smallest eigenvalue is $\Psi_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Let's apply J^+ to this vector repeatedly: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to \sqrt{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. We get all the eigenvectors (not normalized, but still valid eigenvectors) and when we run out of eigenvectors, we get zero vectors. Alternatively, we can start with the eigenvector with the largest eigenvalue $\Psi_{+1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and apply J^- repeatedly: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \sqrt{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to 2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Electroman strolls by and says: "That's just like a shift register!". See the diagram for examples in two, three, and four dimensions.

The above example shows that J^+ and J^- do not keep the eigenvectors in their normalized form. This can be fixed by including the following normalization factors: $J^+\Psi_m = \sqrt{j(j+1) - m(m+1)}\Psi_{m+1}$ and $J^-\Psi_m = \sqrt{j(j+1) - m(m-1)}\Psi_{m-1}$, where j is the maximum value of m [GTNut, Ch. IV.2; PfS, Ch. 3.6.1]. In our example we have $m = -1, 0, 1$ and thus $j = 1$; the raising operations with the normalization factor included become $J^+\Psi_{-1} = \sqrt{2}\Psi_0$ and $J^+\Psi_0 = \sqrt{2}\Psi_{+1}$; similarly, the lowering operations become $J^-\Psi_{+1} = \sqrt{2}\Psi_0$ and $J^-\Psi_0 = \sqrt{2}\Psi_{-1}$. Now, Ψ_m remains normalized!

It is interesting to compare the eigenvalues of the generators of U(1) and SU(2). For a d-dimensional representation of U(1) with $d > 1$, the eigenvalues must be integer multiples of a common value (for $d = 1$, any value is fine). The reason for this condition is that the circular topology of the *group* imposes a periodicity constraint. The (trivial) Lie *algebra* of U(1) has nothing to say about these eigenvalues. In contrast, the Lie algebra of SU(2) demands, as we have just seen, that the eigenvalues are spaced apart by 1.0. This follows entirely from the local structure (Lie algebra); the global structure (topology) of the SU(2) group was not used.