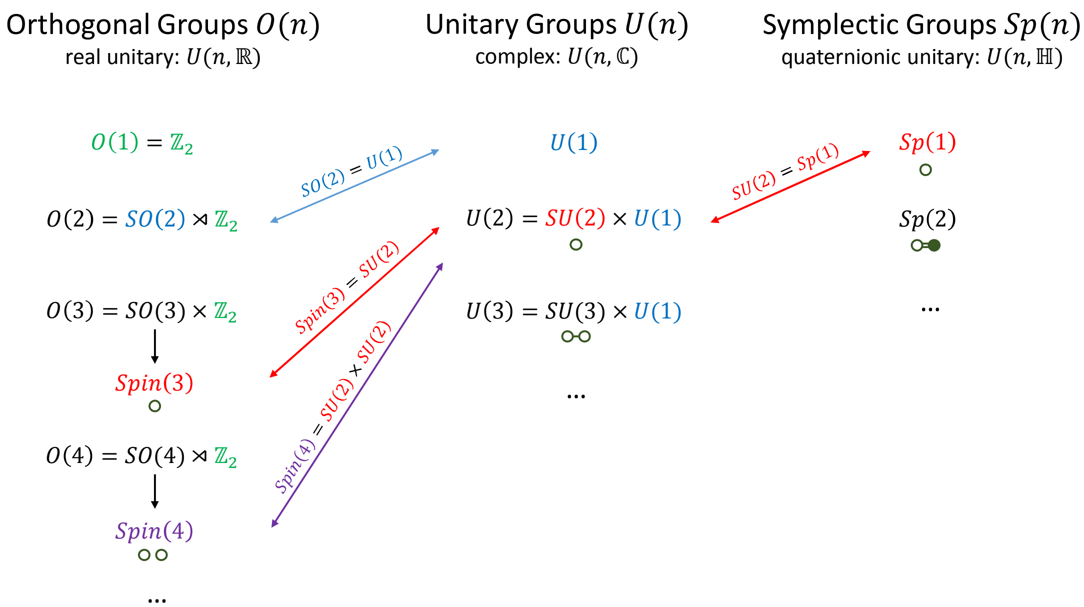
5. Appendices

5.1 Low-Dimensional Lie Groups and their Relationships



Orthogonal Groups O(n). The orthogonal group O(n) consists of all the linear transformations that preserve the Euclidean inner product (dot product) of two real n-dimensional vectors. Topologically, the group O(n) consists of two disconnected components: one with the reflective and one with the non-reflective transformations. Thus, only half of the elements can be reached in a continuous manner when starting from the identity element. By restricting the transformations to only those with determinant plus one (non-reflective elements), we are led to the subgroup SO(n). Topologically, this group consists of only one component, that is, the group manifold is *connected*. Essentially, to get from O(n) to SO(n) we "divided out" the cyclic group of order two $\mathbb{Z}_2$ (multiplicative group with elements $+1$ and -1). We can thus write O(n) as the "product" SO(n) times $\mathbb{Z}_2$ (the direct product if n is odd and the semidirect product if n is even; see https://math.stackexchange.com/questions/1055363/is-o2-really-not-isomorphic-to-so2-times-1-1?rq=1). Dividing out $\mathbb{Z}_2$ also has the advantage that the result is free of nontrivial normal subgroups, that is, SO(n) is a *simple group*. Although SO(n) is topologically connected it is still not *simply connected*. To be simply connected it is necessary that all closed loops in the group manifold can be contracted to a point in a continuous manner. Interestingly, it is possible to find a larger group, Spin(n), known as the covering group, which has the same Lie algebra as SO(n) and therefore is locally isomorphic to SO(n), that is *simply connected* (for $n > 2$). (Note that the covering group of a simple group is not necessarily simple anymore.) The left column of the diagram shows the groups O(n) for $n = 1, \ldots, 4$ together with the derived groups SO(n) and Spin(n).

Unitary Groups U(n). The unitary group U(n) consists of all the linear transformations that preserve the Hermitian inner product of two complex n-dimensional vectors. Topologically, U(n) is connected. U(n) is *not* a simple group (for $n > 1$), but by restricting the transformations to only those with unit determinant, we are led to the subgroup SU(n), which is a *simple group*. Essentially, to get from U(n) to

SU(n) we “divided out” the subgroup U(1). We can thus write U(n) as the “product” SU(n) times U(1) (more specifically, U(n) = (SU(n)/$\mathbb{Z}_n$) × U(1) [GTNut, p. 253]). Topologically, SU(n) is *simply connected* and thus no further polishing is necessary. (Incidentally, U(1) is *not* simply connected.) The middle column of the diagram shows the groups U(n) for $n = 1, 2, 3$ together with the derived groups SU(n).

Symplectic Groups Sp(n). The compact symplectic group Sp(n) consists of all the linear transformations that preserve the inner product of two quaternionic n-dimensional vectors. Sp(n) is a *simple group* (no nontrivial normal subgroups) and it is *simply connected* (loops can be continuously contracted to a point). These groups are beautiful gems straight out of the box! The right column of the diagram shows the groups Sp(n) for $n = 1, 2$. (Caution! Different authors use different naming conventions: what we call Sp(n) is sometimes called Sp(2n) or USp(2n).)

Relationships. Interesting relationships between the groups shown in the diagram exist:

- Groups SO(2) and U(1) are isomorphic. Both groups describe 2-dimensional rotations.
- Groups Spin(3), SU(2), and Sp(1) are isomorphic. All three groups describe 3-dimensional rotations of spinorial objects. The Dynkin diagram for all three groups (or rather their algebras) is a small circle. (Dynkin diagrams are composed of one small circle per simple root vector [the number of circles equals the rank of the algebra], the lines in between the circles indicate the angle between the root vectors [no line for 90°], and the color of the circles [black or white] indicates the length of the root vectors [GTNut, Ch. VI.5].)
- Group Spin(4) has two identical normal subgroups and can be written as the direct product Spin(4) = SU(2) × SU(2). This relationship has implications for the existence of two types of spinors in our 4D space-time. The Dynkin diagram of Spin(4) (or rather its algebra: spin(4) = so(4)) consists of two disconnected circles, symbolizing the two identical subgroups.
- Groups Spin(5) and Sp(2) are isomorphic (not shown in the diagram). [I don’t know if this relationship plays a role in physics.]

All the groups shown in the diagram are topologically *compact*, which means that their volume is finite, that is, they don’t go off to infinity [RtR, Ch. 12.6]. What about *non-compact* groups? For example, the *Lorentz group* O(1,3), describing rotations and relativistic boosts, is an important non-compact group. While O(4) preserves the Euclidean distance d given by $d^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$, O(1,3) preserves the “space-time distance” τ (called proper time) given by $\tau^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$. These differences in signs (or difference in signature) make the group non-compact. Nevertheless, O(4) and O(1,3) can be understood as two different real forms of O(4,ℂ), the complexification of O(4) [RtR, Ch. 13.8]. Another important non-compact group is the group of (1-dimensional) *translations*, ℝ. This group is the universal cover of U(1) and SO(2). In a way, the translation group is an unwound (and thus simply connected) version of the rotation group.

All the groups shown in the diagram are *continuous*, that is, they are Lie groups. What about *discrete* groups with a finite number of elements? For example, the group of symmetries of the equilateral triangle, D_3, has six elements and the group of symmetries of the square, D_4, has eight elements. In general, the *dihedral group*, D_n, which describes the symmetries of a regular polygon with n vertices, has $2n$ elements. If we let $n \to \infty$, then $D_n \to$ O(2), the group of symmetries of a circle! The *cyclic group*, $\mathbb{Z}_n$, has n elements and describes the non-reflective symmetries of a regular polygon with n vertices. If we let $n \to \infty$, then $\mathbb{Z}_n \to$ SO(2), the group of non-reflective symmetries of a circle!